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Existence for the Neumann stochastic semilinear equations via an optimal control approach

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Abstract

In the present work we prove existence and uniqueness of the solution for the stochastic nonlinear equation with Neumann boundary conditions, under general monotonicity assumptions, motivated by physical applications. To this purpose we shall use an optimal control approach based on the variational principle of Brezis and Ekeland.

Key word: stochastic PDE's, monotone operators, optimal control, variational principle of Brezis and Ekeland

1 Introduction

This work is concerned with the semilinear stochastic equation with Neumann boundary condition

$$\begin{cases} dX(t) - \Delta X(t)\,dt = \sqrt{Q}dW(t), & (0,T)\times\mathcal{O}, \\ \dfrac{\partial X}{\partial n} + \Phi(X) \ni 0, & (0,T)\times\partial\mathcal{O} = \Sigma_T, \\ X(0) = x, & \mathcal{O}, \end{cases} \tag{1}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$, $\dfrac{\partial}{\partial n}$ is the outward normal derivative on the boundary of $\mathcal{O}$, Φ is a maximal monotone graph (possibly multivalued). We assume that $W(t)$ is a cylindrical Wiener process on a stochastic basis $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ taking values in the Hilbert space $L^2(\mathcal{O})$, defined by $W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j$, for all $t \geq 0$, where $\{e_j\}$ is an orthonormal basis in $L^2(\mathcal{O})$ and $\left\{\beta_j\right\}_{j=1}^{\infty}$ is a sequence of mutually independent Brownian motion on the probability space. We also assume that $x \in L^2(\mathcal{O})$.

The operator Q it considered to be linear, continuous, self-adjoint and positive on $L^2(\mathcal{O})$, with finite trace such that $\text{Ker}Q = \{0\}$.

Existence and uniqueness of the solution for this equation was studied so far by standard existence theory only under very restrictive hypotheses, i.e. Φ is assumed to be a maximal monotone operator defined everywhere on $\mathbb{R}$, monotonically increasing and satisfying

$$\begin{aligned} &\Phi(0) = 0, \quad \text{for all } r \in \mathbb{R}, \\ &|\Phi(r)| \leq C_1 |r| + C_2, \quad \text{for all } r \in \mathbb{R}, \\ &(\Phi(r) - \Phi(s))(r - s) \geq C_3 (r - s)^2, \quad \text{for all } r, s \in \mathbb{R}, \end{aligned} \tag{2}$$

for some constants $C_i > 0$, $i = \overline{1,3}$.

Indeed, considering the operator $A : H^1(\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$ defined by

$$(A(y), \psi) = \int_{\mathcal{O}} \langle \nabla y, \nabla \psi \rangle \, dx + \int_{\partial \mathcal{O}} \Phi(\gamma_0(y)) \gamma_0(\psi) \, d\sigma, \quad \text{for all } \psi \in H^1(\mathcal{O}),$$

where

$$\gamma_0 : H^1(\mathcal{O}) \to H^{1/2}(\partial \mathcal{O}) \tag{3}$$

is the trace function. (For a rigorous definition and details see e.g. [9] page 315.)

Consequently, we can rewrite equation (1) as

$$\begin{cases} dX(t) + AX(t) \, dt = \sqrt{Q} dW(t), \\ X(0) = x \end{cases}$$

and check that, under assumptions (2), the operator A satisfies hypotheses from [18]. Another approach for existence of solution for the same equation was recently published in [6].

In the present work we prove existence and uniqueness of the solution for equation (1) under more general assumptions, motivated by physical applications. To this purpose we shall use an optimal control approach based on the method formulated by Brezis and Ekeland in [10] and [11].

The same approach was used in [16] and [17] for deterministic equations with time periodic coefficients.

Concerning stochastic differential equations, similar results were already proved for the semilinear equation with Dirichlet boundary condition and for the porous media equation in [2] and [3], but the case with Neumann boundary conditions is still an open problem.

Physical motivation

An important model which is not covered by assumptions (2) is the temperature control regulated by the temperature flux at the boundary. In this case the operator Φ is multivalued and the standard form is

$$\Phi(x) = \begin{cases} g_1, & \text{if} \quad x < h_1, \\ [g_1, 0], & \text{if} \quad x = h_1, \\ 0, & \text{if} \quad h_1 < x < h_2, \\ [0, g_2], & \text{if} \quad x = h_2, \\ g_2 & \text{if} \quad h_2 < x. \end{cases}$$

Here $[g_1, g_2]$, with $0 \in [g_1, g_2]$, is the closed interval confining the flux of injected heat which can be measured by $\frac{\partial X}{\partial n}$ and h_1 and h_2 are the reference temperature. For details see Example 3.6 from page 31 of [14]. A similar model describes diffusion process through semi-permeable walls.

Another application of equation (1) arise in radiation models and more precisely in models involving Stefan-Boltzmann radiation law. In this case we need an operator of the form

$$\Phi(x) = a\ x^4 + b.$$

See also the black body radiation model.

Preliminaries and notations

For the reader's convenience, we shall recall some definitions and properties concerning lower-semicontinuous convex functions.

Given such a function $\varphi : V \to \overline{\mathbb{R}} = [-\infty, \ +\infty]$, where V is a Banach space, we denote by $\partial\varphi : V \to V'$ the subdifferential of φ, that is defined by

$$\partial\varphi(x) = \{z \in V';\ \varphi(x) \le \varphi(u) + (x - u, z),\ \forall u \in V\}, \quad x \in V,$$

and by $\varphi^* : V' \to \overline{\mathbb{R}}$ its conjugate which is defined by

$$\varphi^*(z) = \sup\{(z, x) - \varphi(x); \quad x \in Y\}, \quad z \in V'.$$

The conjugate φ^* is also lower-semicontinuous and convex.

We recall also the duality relations

$$\begin{aligned} &\varphi(y) + \varphi^*(z) \ge (y, z), \quad \forall y \in V \text{ and } \forall z \in V', \\ &\varphi(y) + \varphi^*(z) = (y, z), \quad iff \quad z \in \partial\varphi(y), \\ &\partial\varphi^* = (\partial\varphi)^{-1}. \end{aligned} \tag{4}$$

We denoted by $(.,.)$ the duality pairing between V and V'. For details see e.g. [8], [19] and [20].

We shall denote by $L^p(\mathcal{O})$ the classical Lebesgue spaces with the usual norm $\|.\|_p$ and by $H^1(\mathcal{O})$, $H^{1/2}(\partial\mathcal{O})$, $W^{m,p}(\mathcal{O})$ the usual Sobolev spaces with the corresponding duals $\left(H^1(\mathcal{O})\right)^*$, $H^{-1/2}(\partial\mathcal{O})$, $W^{-m,p}(\mathcal{O})$, respectively.

If H is a Hilbert space, we denote by $C_W([0,T];\ H) = C_W\left([0,T]; L^2(\Omega;\ H)\right)$ the space of all the continuous functions $X : [0,T] \to L^2(\Omega;\ H)$ which are adapted to the Wiener process W. This space is provided with the norm

$$\|X\|_{C_W([0,T];\ H)} = \left(\sup_{t\in[0,T]} \mathbb{E}\,|X(t)|_H^2\right).$$

We can define similarly $L^p_W([0,T];\ H)$. For details see [12] and [18].

We shall denote by C an independent constant that may change during the computations.

Hypotheses and definition of the solution

We assume that

H$_1$ The operator $\Phi : \mathbb{R} \to 2^{\mathbb{R}}$, is a maximal monotone operator such that

$$D(\Phi) = \mathbb{R} \text{ and } R(\Phi) = \mathbb{R}.$$

Note that the assumption above holds for the operators presented in the physical motivation.

H$_2$ The potential g of the operator Φ verifies

$$g(-r) \leq C_1 g(r) + C_2, \quad \forall r \in \mathbb{R},$$

where $C_1 > 0$. With no loss of generality we may also consider that $g \geq 0$ and therefore we have also that $g^* \geq 0$.

H$_3$ The operator Q from the noise is such that the stochastic convolution $W_Q \in C\left([0,T] \times \overline{\mathcal{O}}\right)$.

Here we considered

$$W_Q(t) = \int_0^t S(t-s)\sqrt{Q}dW(s), \quad t \geq 0,$$

and $S(t-s)$ is the C_0-semigroup generated on $L^2(\mathcal{O})$ by the Laplace operator with Neumann boundary condition.

For sufficient assumptions on Q under which the condition above holds, see Theorem 2.13, page 29 from [13].

Definition 1 *A mild solution to equation (1) is a stochastic adapted process* $X \in C_W\left([0,T];\ \left(H^1(\mathcal{O})\right)^*\right)$ *which satisfies*

$$X(t) = e^{-A_0 t}x - \int_0^t e^{-A_0(t-s)}B(Z)\,ds + W_Q(t), \quad \mathbb{P}\text{-a.s.},\ t \in [0,T],$$

where

$$Z \in L^1([0,T] \times \partial\mathcal{O} \times \Omega) \cap L^2_W\left([0,T];\ H^{-1/2}(\partial\mathcal{O})\right)$$

is such that $Z \in \Phi(X)$ *a.e. on* $[0,T] \times \partial\mathcal{O} \times \Omega$,

$$A_0 : H^1(\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$$

is the Laplace operator with Neumann boundary condition, $e^{-A_0 t}$ *is the* C_0 *semigroup generated by* A_0 *on* $\left(H^1(\mathcal{O})\right)^*$ *and*

$$B : H^{-1/2}(\partial\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$$

is the adjoint of the trace operator γ_0 *introduced in (3).*

2 The optimal control formulation

The first step is to rewrite the stochastic differential equation (1) as a random differential equation.

To this purpose we consider first

$$\begin{cases} dW_Q(t) - \Delta W_Q(t)\,dt = \sqrt{Q}dW(t), & \mathcal{O}\times(0,T), \\ \dfrac{\partial W_Q}{\partial n} = 0, & \partial\mathcal{O}\times(0,T), \\ W_Q(0) = 0, & \mathcal{O}. \end{cases} \tag{5}$$

It is well known that the restriction to $H^2(\mathcal{O})$ of the Laplace operator A_0 generates a C_0- semigroup on $L^2(\mathcal{O})$, which shall be denoted by $S(t)$.

By classical existence theory we have that equation (5) has a unique solution (the stochastic convolution) of the form

$$W_Q(t) = \int_0^t S(t-s)\sqrt{Q}dW(s), \quad t \geq 0,$$

such that $W_Q \in C\left([0,T]\times\overline{\mathcal{O}}\right)$, $\mathbb{P}-a.s.$.

For each $\omega\in\Omega$ fixed, we can take the difference between (1) and (5) and get

$$\begin{cases} d(X(t) - W_Q(t)) - \Delta(X(t) - W_Q(t))\,dt = 0, & (0,T)\times\mathcal{O}, \\ \dfrac{\partial(X - W_Q)}{\partial n} + \Phi(X) \ni 0, & (0,T)\times\partial\mathcal{O} = \Sigma_T, \\ (X - W_Q)(0) = x, & \mathcal{O}. \end{cases} \tag{6}$$

We denote $Y = X - W_Q$ and we can rewrite (6) in the equivalent form

$$\begin{cases} \dfrac{\partial Y}{\partial t} - \Delta Y = 0, & (0,T)\times\mathcal{O}, \\ \dfrac{\partial Y}{\partial n} + Z = 0, & \Sigma_T, \\ Y(0) = x, & \mathcal{O}, \end{cases}$$

where

$$g(\gamma_0(Y + W_Q)) + g^*(Z) = \gamma_0(Y + W_Q)Z, \quad a.e. \text{ on } \Sigma_T.$$

Indeed, according to (4), the relation above is verified only if

$$Z \in \Phi(\gamma_0(Y + W_Q)), \text{ a.e. on } (0,T)\times\partial\mathcal{O}\times\Omega.$$

We construct the following optimal problem

$$\begin{aligned} & \qquad Minimize \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbf{P}) \\ I(Y,Z) &= \int_0^T \int_{\partial\mathcal{O}} \left\{ g\left(\gamma_0\left(Y+W_Q\right)\right) + g^*\left(Z\right) - \gamma_0\left(Y+W_Q\right) Z \right\} d\sigma dt \end{aligned}$$

where

$$\gamma_0\left(Y\right) \in L^1\left(\Sigma_T\right), Z \in L^1\left([0,T]\times\partial\mathcal{O}\right) \cap L^2\left([0,T];\ H^{-1/2}\left(\partial\mathcal{O}\right)\right)$$

are subject to

$$\begin{cases} \dfrac{\partial Y}{\partial t} - \Delta Y dt = 0, & (0,T)\times\mathcal{O}, \\ \dfrac{\partial Y}{\partial n} + Z = 0, & \Sigma_T, \\ Y\left(0\right) = x, & \mathcal{O}, \end{cases}$$

and to the state contrainte $Y \in L^2\left((0,T);\ H^1\left(\mathcal{O}\right)\right)$.

Following an idea similar to the Brezis Ekeland variational principle, we can easily check that equation (6) has a solution if and only if the optimal problem $(\mathbf{P})$ has an optimal pair $(Y^*,\ Z^*)$ and $I\left(Y^*,\ Z^*\right) = 0$.

Since we shall use a L^1- approach, the last term of $I(Y,Z)$ might not be well defined. For this reason we shall rewrite $(\mathbf{P})$ in a more convenient form, i.e.

$$\begin{aligned} & \qquad Minimize \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbf{P}') \\ I'(Y,Z) &= \int_0^T \int_{\partial\mathcal{O}} \left\{ g\left(\gamma_0\left(Y+W_Q\right)\right) + g^*\left(Z\right) - \gamma_0\left(W_Q\right) Z \right\} d\sigma dt \\ & \quad + \int_0^T \int_{\mathcal{O}} \left|\nabla Y\right|^2 d\xi dt + \frac{1}{2}\left\|Y\left(T\right)\right\|_2^2 - \frac{1}{2}\left\|x\right\|_2^2, \end{aligned}$$

where

$$\gamma_0\left(Y\right) \in L^1\left(\Sigma_T\right), Z \in L^1\left([0,T]\times\partial\mathcal{O}\right) \cap L^2\left([0,T];\ H^{-1/2}\left(\partial\mathcal{O}\right)\right)$$

are subject to

$$\begin{cases} \dfrac{\partial Y}{\partial t} - \Delta Y dt = 0, & (0,T)\times\mathcal{O}, \\ \dfrac{\partial Y}{\partial n} + Z = 0, & \Sigma_T, \\ Y\left(0\right) = x, & \mathcal{O}, \end{cases}$$

and to the state contrainte $Y \in L^2\left((0,T);\ H^1\left(\mathcal{O}\right)\right)$.

3 The main result

We can now formulate the main result of this paper which holds under the hypotheses presented in introduction.

Theorem 2 *For each $x \in L^2(\mathcal{O})$, there is a unique mild solution to equation (1) in the sense of Definition 1, such that*

$$X \in L^2_W \left([0,T];\ H^1(\mathcal{O})\right)$$

and

$$g\left(\gamma_0 (Y + W_Q)\right),\ g^*(Z) \in L^1(\Omega \times [0,T] \times \partial\mathcal{O}),$$

where $Z \in \Phi(X)$ a.e. on $[0,T] \times \partial\mathcal{O} \times \Omega$.

In order to prove the result above, we need the following lemmas.

Lemma 3 *The optimal control problem ($\mathbf{P}'$) has at least a optimal pair.*

Proof of Lemma 3. We know by classical theory that for

$$Z \in L^2\left(0,T;\ H^{-1/2}(\partial\mathcal{O})\right)$$

equation

$$\begin{cases} \dfrac{\partial Y}{\partial t} - \Delta Y dt = 0, & (0,T) \times \mathcal{O}, \\ \dfrac{\partial Y}{\partial n} + Z = 0, & \Sigma_T, \\ Y(0) = x, & \mathcal{O}, \end{cases} \tag{7}$$

has a unique solution $Y \in C\left([0,T];\ \left(H^1(\mathcal{O})\right)^*\right) \cap L^2\left(0,T\ ; H^1(\mathcal{O})\right)$, of the form

$$Y(t) = e^{-A_0 t}x - \int_0^t e^{-A_0(t-s)} B(Z)\, ds \tag{8}$$

where the operators A_0 and B are as in Definition 1.

A similar method is presented in Example 2 from Section 4.3 of [8] for the restriction of A_0 to $H^2(\mathcal{O})$ which generates a C_0- semigroup on $L^2(\mathcal{O})$. For the reader's convenience, we shall sketch the main ideas, adapted to our case, which needs A_0 as the generator of a C_0- semigroup on $\left(H^1(\mathcal{O})\right)^*$.

Indeed, (8) follows by Proposition 4.39 from [8]. For assumption *iii*) which is necessary for the proposition mentioned before, we need to check that there exist a function $\varphi \in L^1(0,T)$ such that

$$\left\|e^{A_0 t} B\right\|_{L\left(H^{-1/2}(\partial\mathcal{O}),\ (H^1(\mathcal{O}))^*\right)} \le \varphi(t), \quad \forall t \in [0,T]. \tag{9}$$

This follows from

$$\int_0^t \|y(s)\|^2_{(H^1(\mathcal{O}))^*}\, ds \le \frac{1}{2}\|y_0\|^2_{(H^1(\mathcal{O}))^*} \tag{10}$$

where $y(t) = e^{-A_0 t}x$ is the solution to

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = 0, && (0,T)\times\mathcal{O}, \\ &\frac{\partial y}{\partial n} = 0, && \Sigma_T, \\ &y(0) = y_0, && \mathcal{O}, \end{aligned}$$

Relation (10) was obtained by using the chain differentiation rule (see e.g. Lemma 4.1 from page 128 of [5]).

More precisely, we obtain (9) by using (10) as follows

$$\begin{aligned} \left\|e^{A_0 t}B\right\|_{L\left(H^{-1/2}(\partial\mathcal{O}),\ (H^1(\mathcal{O}))^*\right)} &\le \sup_{\|z\|_{H^{-1/2}(\partial\mathcal{O})}\le 1} \left\|e^{A_0 t}B(z)\right\|_{(H^1(\mathcal{O}))^*} \\ &\le \sup_{\|z\|_{H^{-1/2}(\partial\mathcal{O})}\le 1} \|B(z)\|_{(H^1(\mathcal{O}))^*}\varphi(t) \\ &\le C\varphi(t), \end{aligned}$$

and because $B : H^{-1/2}(\partial\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$ is linear and continuous as its adjoint γ_0.

In order to get existence of the optimal pair we need first to prove that the cost functional

$$\begin{aligned} I'(Y,Z) = &\int_0^T\int_{\partial\mathcal{O}} \{g(\gamma_0(Y+W_Q)) + g^*(Z) - \gamma_0(W_Q)Z\}\, d\sigma dt \qquad (11) \\ &+ \int_0^T\int_{\mathcal{O}} |\nabla Y|^2\, d\xi dt + \frac{1}{2}\|Y(T)\|_2^2 - \frac{1}{2}\|x\|_2^2, \end{aligned}$$

is bounded from below. For similar arguments applied to the porous media case, see [2] and [7].

First we note that

$$\left|\int_0^T\int_{\partial\mathcal{O}} \gamma_0(W_Q)Z d\sigma dt\right| \le \delta\int_0^T\int_{\partial\mathcal{O}} |Z|\, d\sigma dt \tag{12}$$

where $\delta = \sup_{t\in[0,T]} |W_Q|_{L^\infty(\partial\mathcal{O})} < \infty$.

On the other hand, since $D(\Phi) = \mathbb{R}$ we have that $\lim_{|r|\to\infty}\frac{g^*(r)}{|r|} = \infty$, and then, for each $N > 1$ there is $C_N > 0$ such that

$$g^*(Z) \ge 2N\delta|Z| \quad \textit{a.e. on } \{(t,\xi)\in\Sigma_T;\ Z(t,\xi) > C_N\}.$$

Then we have that

$$\int_0^T \int_{\partial\mathcal{O}} |Z|\, d\sigma dt = \iint\limits_{|Z(t,\xi)|>C_N} |Z|\, d\sigma dt + \iint\limits_{|Z(t,\xi)|\le C_N} |Z|\, d\sigma dt \qquad (13)$$
$$\le \frac{1}{2N\delta} \int_0^T \int_{\partial\mathcal{O}} g^*(Z)\, d\sigma dt + C_N |\Sigma_T|,$$

where $|\Sigma_T|$ is the measure of Σ_T.

Going back to (12) we obtain

$$\left| \int_0^T \int_{\partial\mathcal{O}} \gamma_0(W_Q)\, Z d\sigma dt \right| \le \frac{1}{2} \int_0^T \int_{\partial\mathcal{O}} g^*(Z)\, d\sigma dt + \delta C_N |\Sigma_T|.$$

Keeping in mind that $g \ge 0$ and $g^* \ge 0$ we obtain that (11) is bounded from below.

Consequently, we can construct a minimizing sequence $(Y_n, Z_n)_n$ that verifies

$$\inf I'(Y,Z) \le \int_0^T \int_{\partial\mathcal{O}} \{g(\gamma_0(Y_n + W_Q)) + g^*(Z_n) - \gamma_0(W_Q)\, Z_n\}\, d\sigma dt \qquad (14)$$
$$+ \int_0^T \int_{\mathcal{O}} |\nabla Y_n|^2\, d\xi dt + \frac{1}{2}\|Y_n(T)\|_2^2 - \frac{1}{2}\|x\|_2^2 \le \inf I'(Y,Z) + \frac{1}{n}$$

and is subject to (7).

We obtain that $(Y_n)_n$ is weakly compact in $L^2\left(0,T;\ H^1(\mathcal{O})\right)$ and, via the Dunford Pettis theorem, we have that $(Z_n)_n$ is weakly compact in $L^1(\Sigma_T)$. Indeed, we have that

$$\int_0^T \int_{\partial\mathcal{O}} g^*(Z_n)\, d\sigma dt \le C\left(1 + \|x\|_2^2\right),$$

and by (13) we get that

$$\int_0^T \int_{\partial\mathcal{O}} |Z_n|\, d\sigma dt \le C\left(1 + \|x\|_2^2\right).$$

The equi-integrability of $(Z_n)_n$ follows by arguing as in Proposition 2.12 from [4] or Theorem 2.2 from [2] and then we can apply the Dunford Pettis theorem.

This leads to

$$\begin{aligned} &Y_n \to Y^*, && \text{weakly in } L^2\left(0,T;\ H^1(\mathcal{O})\right), \\ &\gamma_0(Y_n) \to \gamma_0(Y^*), && \text{weakly in } L^1(\Sigma_T), \\ &Z_n \to Z^*, && \text{weakly in } L^1(\Sigma_T). \end{aligned} \qquad (15)$$

Since g and g^* are lower semicontinuous and convex in $\mathbb{R}$, we have by [20] that the corresponding convex integrals have the same properties in $L^1(\Sigma_T)$

and, consequently, they are lower semicontinuous also in the weak topology of $L^1(\Sigma_T)$.

Now we can pass to the limit in (14) and get a solution to the optimal control problem, i.e.

$$\inf I'(Y,Z) = \int_0^T \int_{\partial\mathcal{O}} \{g(\gamma_0(Y^* + W_Q)) + g^*(Z^*) - \gamma_0(W_Q) Z^*\} \, d\sigma dt$$

$$+ \int_0^T \int_{\mathcal{O}} |\nabla Y^*|^2 \, d\xi dt + \frac{1}{2} \|Y^*(T)\|_2^2 - \frac{1}{2} \|x\|_2^2,$$

and this conclude the proof of Lemma 3. ■

Lemma 4 *The optimal solution* (Y^*, Z^*) *to problem* ($\mathbf{P}'$) *is also solution to the optimization problem* ($\mathbf{P}$).

Proof of Lemma 4. From the formulation of problem ($\mathbf{P}'$) we know that (Y^*, Z^*) are such that

$$Z^*, \ \gamma_0(Y^*) \in L^1(\Sigma_T), \quad Y^*(T) \in L^2(\mathcal{O}),$$

and

$$Y^* \in C\left([0,T];\ L^1(\mathcal{O})\right) \cap L^2\left(0,T;\ H^1(\mathcal{O})\right)$$

is a mild solution to equation (7) such that

$$g(\gamma_0(Y^* + W_Q)), \ g^*(Z^*) \in L^1(\Sigma_T).$$

First we shall check that

$$\gamma_0(Y^*) Z^* \in L^1(\Sigma_T). \tag{16}$$

By using assumption H_2 we can easily see that

$$g(-\gamma_0(Y^* + W_Q)) \in L^1(\Sigma_T),$$

and then, by the conjugacy formulae (4), we obtain that

$$-g(-\gamma_0(Y^* + W_Q)) - g^*(Z^*) \leq \gamma_0(Y^* + W_Q) Z^* \leq g(\gamma_0(Y^* + W_Q)) + g^*(Z^*)$$

and consequently, (16) follows directly.

Next we shall prove that

$$\frac{1}{2} \|Y^*(T)\|_2^2 - \frac{1}{2} \|x\|_2^2 + \int_0^T \int_{\mathcal{O}} |\nabla Y^*|^2 \, d\xi dt + \int_0^T \int_{\partial\mathcal{O}} \gamma_0(Y^*) Z^* d\sigma dt = 0. \tag{17}$$

From the definition of the problem ($\mathbf{P}'$) we know that $Z^* \in L^1(\Sigma_T)$ is such that the solution of equation

$$\begin{cases} \dfrac{\partial Y^*}{\partial t} - \Delta Y^* = 0, & (0,T) \times \mathcal{O}, \\ \dfrac{\partial Y^*}{\partial n} + Z^* = 0, & \Sigma_T, \\ Y^*(0) = x, & \mathcal{O}, \end{cases} \tag{18}$$

verifies $Y^* \in C\left([0,T];\ L^1(\mathcal{O})\right) \cap L^2\left(0,T;\ H^1(\mathcal{O})\right)$ and $Y^*(T) \in L^2(\mathcal{O})$.

Since

$$\frac{\partial Y^*}{\partial n} + Z^* = 0 \text{ on } \Sigma_T$$

and $Y^* \in L^2\left(0,T;\ H^1(\mathcal{O})\right)$ we have by Theorem 7.39, page 234 from [1] that

$$Z^* \in L^2\left(0,T;\ H^{-1/2}(\partial\mathcal{O})\right).$$

See also [15] for similar arguments.

We recall the trace operator

$$\gamma_0 : H^1(\mathcal{O}) \longrightarrow H^{1/2}(\partial\mathcal{O}),$$

and its adjoint

$$B : H^{-1/2}(\partial\mathcal{O}) \longrightarrow \left(H^1(\mathcal{O})\right)^*,$$

i.e.,

$$_{(H^1(\mathcal{O}))^*}\langle B(u), v\rangle_{H^1(\mathcal{O})} \;=\; _{H^{-1/2}(\partial\mathcal{O})}\langle u, \gamma_0(v)\rangle_{H^{1/2}(\partial\mathcal{O})}.$$

Then equation (18) can be rewritten as

$$\begin{cases} \dfrac{\partial Y^*}{\partial t} + A_0(Y^*) + B(Z^*) = 0, & (0,T) \\ Y^*(0) = x \end{cases} \tag{19}$$

where $A_0 : H^1(\mathcal{O}) \longrightarrow \left(H^1(\mathcal{O})\right)^*$ is the Laplace operator with Neumann homogeneous boundary conditions.

We apply to (19) the operator $(1+\varepsilon A_0)^{-1} : \left(H^1(\mathcal{O})\right)^* \longrightarrow H^1(\mathcal{O})$ and denote

$$Y_\varepsilon^* = (1+\varepsilon A_0)^{-1} Y^* \quad \text{and} \quad B(Z^*)_\varepsilon = (1+\varepsilon A_0)^{-1} B(Z^*).$$

It is well known that $Y_\varepsilon^* \to Y^*$ and $B(Z^*)_\varepsilon \to B(Z^*)$ as $\varepsilon \to 0$ in the spaces where they belong (i.e. in $H^1(\mathcal{O})$ and $\left(H^1(\mathcal{O})\right)^*$ respectively).

On the other hand we can take the duality product between $H^1(\mathcal{O})$ and $\left(H^1(\mathcal{O})\right)^*$ of equation

$$\begin{cases} \dfrac{\partial Y_\varepsilon^*}{\partial t} + A_0(Y_\varepsilon^*) + B(Z^*)_\varepsilon = 0, & (0,T) \\ Y_\varepsilon^*(0) = (1+\varepsilon A_0)^{-1} x. \end{cases}$$

with Y_ε^*, and, keeping in mind that ${}_{(H^1(\mathcal{O}))^*}\langle .\ ,\ .\rangle_{H^1(\mathcal{O})} = \langle .\ ,\ .\rangle_2$, we have that

$$\frac{1}{2}\left\|Y_\varepsilon^*(T)\right\|_2^2 - \frac{1}{2}\left\|(1+\varepsilon A_0)^{-1}x\right\|_2^2 + \int_0^T\int_{\mathcal{O}}\left|\nabla Y_\varepsilon^*\right|^2 d\xi dt + \int_0^T\int_{\mathcal{O}} B(Z^*)_\varepsilon Y_\varepsilon^* d\xi dt = 0.$$

Letting $\varepsilon \to 0$, we obtain that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathcal{O}} B(Z^*)_\varepsilon Y_\varepsilon^* d\xi dt = -\frac{1}{2}\left\|Y^*(T)\right\|_2^2 + \frac{1}{2}\left\|x\right\|_2^2 - \int_0^T\int_{\mathcal{O}}\left|\nabla Y^*\right|^2 d\xi dt. \tag{20}$$

We also know that

$$Y_\varepsilon^* \to Y^*, \text{ strongly in } H^1(\mathcal{O})$$

and

$$B(Z^*)_\varepsilon \to B(Z^*), \text{ strongly in } \left(H^1(\mathcal{O})\right)^*$$

in each $t \in [0,T]$ as $\varepsilon \to 0$, and, by using the fact that $(1+\varepsilon A_0)^{-1}$ is a contraction and B is linear and continuous, we get that

$$\begin{aligned}\left|\int_{\mathcal{O}} B(Z^*)_\varepsilon Y_\varepsilon^* d\xi\right| &\leq \left\|B(Z^*)_\varepsilon\right\|_{(H^1(\mathcal{O}))^*}\left\|Y_\varepsilon^*\right\|_{H^1(\mathcal{O})} \\ &\leq \left\|Z^*\right\|_{H^{-1/2}(\partial\mathcal{O})}\left\|Y^*\right\|_{H^1(\mathcal{O})}.\end{aligned}$$

Keeping in mind that $Z^* \in L^2\left(0,T;\ H^{-1/2}(\partial\mathcal{O})\right)$ and $Y^* \in L^2\left(0,T;\ H^1(\mathcal{O})\right)$ we can pass to the limit in (20) by using the Lebesgue dominated convergence theorem for the integral with respect to t and we obtain (17).

Finally, by replacing (17) in $(\mathbf{P}')$ we can conclude the proof of Lemma 4. ■

Proof of the main result.

In order to prove existence of the solution for equation (1) it is sufficient to show that $I(Y^*, Z^*) = 0$. To this purpose we can use the duality theorem for optimal control problem (see for instance Theorem 4.16 from [8]), i.e.

$$I'(Y^*, Z^*) + \min(P^*) = 0,$$

where (P^*) is the dual optimization problem corresponding to (P) or equivalently to (P').

Recall that

$$\begin{aligned} I'(Y^*, Z^*) &= \int_0^T\left[\int_{\partial\mathcal{O}}\left\{g\left(\gamma_0(Y^* + W_Q)\right) + g^*(Z^*) - Z^*\gamma_0(W_Q)\right\}d\sigma\right]dt \\ &\quad + \int_0^T\int_{\mathcal{O}}\left|\nabla Y^*\right|^2 d\xi dt + \frac{1}{2}\left\|Y^*(T)\right\|_2^2 - \frac{1}{2}\left\|Y^*(0)\right\|_2^2 \\ &\overset{denote}{=} \int_0^T L(Y^*,\ Z^*)\,dt + l(Y^*(T),\ Y^*(0)). \end{aligned}$$

We can compute the explicit form of the dual problem by using the following constructions.

The functional of the dual problem is

$$I^*(p,q) = \int_0^T M\left(\gamma_0\left(p\right), q\right) dt + m(p(T), p(0)), \tag{21}$$

where

$$M\left(q, \gamma_0\left(p\right)\right) = \sup_{(u,v)\in H^1\times H^{-1/2}} \left\{ {}_{H^1}\left(u,q\right)_{(H^1)^*} + {}_{H^{-1/2}}\left(v, \gamma_0\left(p\right)\right)_{H^{1/2}} - L(u,v) \right\},$$

for $p \in H^1\left(\mathcal{O}\right)$ and $q \in \left(H^1\left(\mathcal{O}\right)\right)^*$ and

$$m(p(T),\ p(0)) = l^*(p(T),\ \ -p(0)),$$

and $I^*(p,q)$ is subject to

$$\begin{cases} \dfrac{\partial p}{\partial t} + \Delta p + q = 0, \quad \text{on } (0,T)\times\mathcal{O}, \\ \\ \dfrac{\partial p}{\partial n} = 0, \quad \text{on } (0,T)\times\partial\mathcal{O}, \end{cases} \tag{22}$$

such that

$$\left(q, \gamma_0\left(p\right)\right) \in \partial L\left(Y^*,\ Z^*\right),$$

$$(p(T), -p(0)) \in \partial l(Y^*\left(T\right),\ Y^*\left(0\right)).$$

For details see e.g. Section 4.1.8 from [8], especially the proof of Theorem 4.5 and Theorem 4.16. Keep in mind that, in our case, the sign of the operator B is changed with respect to the mentioned theory and that implies the other sign modifications which can be seen above.

We shall first compute

$$\begin{aligned} M\left(q, \gamma_0\left(p\right)\right) &= \sup_{u\in H^1(\mathcal{O})} \left\{ {}_{H^1}\left(u,q\right)_{(H^1)^*} - \tilde{I}_g(u) \right\} \\ &\quad + \sup_{v\in H^{-1/2}(\mathcal{O})} \left\{ {}_{H^{-1/2}}\left(v, \gamma_0\left(p\right)\right)_{H^{1/2}} - \tilde{I}_{g^*}(v) \right\} \\ &\overset{denote}{=} \left(\tilde{I}_g\right)^*(q) + \left(\tilde{I}_{g^*}\right)^*\left(\gamma_0\left(p\right)\right) \end{aligned}$$

where

$$\begin{aligned} \tilde{I}_g(u) &= \int_{\partial\mathcal{O}} g\left(\gamma_0\left(u + W_Q\right)\right) d\sigma + \int_{\mathcal{O}} \left|\nabla u\right|^2 d\xi \\ \tilde{I}_{g^*}(v) &= \int_{\partial\mathcal{O}} \left(g^*\left(v\right) - v\gamma_0\left(W_Q\right)\right) d\sigma. \end{aligned}$$

We can easily see that

$$\begin{aligned}\left(\tilde{I}_{g^*}\right)^*(\gamma_0(p)) &= \sup_{v\in H^{-1/2}(\mathcal{O})}\left\{{}_{H^{-1/2}}(v,\gamma_0(p)+\gamma_0(W_Q))_{H^{1/2}}-\int_{\partial\mathcal{O}} g^*(v)\,d\sigma\right\}\\ &= \int_{\partial\mathcal{O}} g(\gamma_0(p+W_Q))\,d\sigma,\end{aligned}$$

and then

$$M(q,\gamma_0(p)) = \left(\tilde{I}_g\right)^*(q)+\int_{\partial\mathcal{O}} g(\gamma_0(p+W_Q))\,d\sigma. \tag{23}$$

On the other hand we have that

$$\begin{aligned} m(p(T),\ p(0)) &= l^*(p(T),\ -p(0)) \qquad (24)\\ &= \sup_{(a,b)\in L^2(\mathcal{O})\times L^2(\mathcal{O})}\left\{(a,p(T))_2+(b,-p(0))_2-\frac{1}{2}\|a\|_2^2+\frac{1}{2}\|b\|_2^2\right\}\\ &= \frac{1}{2}\|p(T)\|_2^2-\frac{1}{2}\|p(0\|_2^2,\end{aligned}$$

since

$$-\varphi^*(r) = (-\varphi)^*(-r) \quad \text{for} \quad \varphi: L^2(\mathcal{O})\to\mathbb{R}, \quad \varphi(b)=\frac{1}{2}\|b\|_2^2.$$

Going back to (21) and replacing (23) and (24) we obtain that

$$\begin{aligned} I^*(p,q) &= \int_0^T\left(\left(\tilde{I}_g\right)^*(q)+\int_{\partial\mathcal{O}} g(\gamma_0(p+W_Q))\,d\sigma\right)dt \qquad (25)\\ &\quad +\frac{1}{2}\|p(T)\|_2^2-\frac{1}{2}\|p(0\|_2^2.\end{aligned}$$

From (22) we can easily see that

$$\frac{1}{2}\|p(T)\|_2^2-\frac{1}{2}\|p(0\|_2^2 = \int_0^T\int_{\mathcal{O}}\left(|\nabla p|^2-pq\right)d\xi dt,$$

and then, by replacing in (25), we get

$$\begin{aligned} I^*(p,q) &= \int_0^T\left(\tilde{I}_g\right)^*(q)\,dt+\int_0^T\left(\int_{\partial\mathcal{O}} g(\gamma_0(p+W_Q))\,d\sigma+\int_{\mathcal{O}}|\nabla p|^2\,d\xi\right)dt\\ &\quad -\int_0^T\int_{\mathcal{O}} pq\,d\xi dt\\ &= \int_0^T\left(\left(\tilde{I}_g\right)^*(q)+\tilde{I}_g(p)-(p,q)_{L^2(\mathcal{O})}\right)dt \geq 0.\end{aligned}$$

Consequently

$$\min(P^*)\geq 0$$

and then $I'(Y^*, Z^*) = 0$ (equivalent with $I(Y^*, Z^*) = 0$), i.e.

$$\int_0^T \int_{\partial\mathcal{O}} \{g(\gamma_0(Y^* + W_Q)) + g^*(Z^*) - \gamma_0(Y^* + W_Q) Z^*\} d\sigma dt = 0.$$

Finally, by (4) we get that

$$Z^* \in \Phi(\gamma_0(Y^* + W_Q)), \text{ a.e. on } (0,T) \times \partial\mathcal{O} \times \Omega.$$

Taking into account that $X = Y + W_Q$ and also the relations (14) and (15) we conclude the proof of existence.

Uniqueness of the solution follows directly from uniqueness of the solution to equation (6) which is immediate from the monotonicity of Φ. This concludes the proof of the main result. ■

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