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The isotropic Cosserat shell model including terms up to $O(h^5)$
Part II: Existence of minimizers

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The isotropic Cosserat shell model including terms up to $O(h^5)$ Part II: Existence of minimizers

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Abstract

We show the existence of global minimizers for a geometrically nonlinear isotropic elastic Cosserat 6-parameter shell model. The proof of the main theorem is based on the direct methods of the calculus of variations using essentially the convexity of the energy in the nonlinear strain and curvature measures. We first show the existence of the solution for the theory including $O(h^5)$ terms. The energy allows us to show the coercivity terms up to order $O(h^5)$ and the convexity of the energy. Secondly, we consider only that part of the energy including $O(h^3)$ terms. In this case the obtained minimization problem is not the same as that previously considered in the literature, since the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem and additional bending-curvature and curvature terms are present. While in the theory including $O(h^5)$ the conditions on the thickness h are those considered in the modelling process and they are independent of the constitutive parameter, in the $O(h^3)$-case the coercivity is proven under some more restrictive conditions under the thickness h.

Keywords: geometrically nonlinear Cosserat shell, 6-parameter resultant shell, in-plane drill rotations, thin structures, dimensional reduction, wryness tensor, dislocation density tensor, isotropy, calculus of variations, uniform convexity

Contents

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1 Introduction

Shell and plate theories are intended for the study of thin bodies, i.e. bodies in which the thickness in one direction is much smaller than the dimensions in the other two orthogonal directions. In this paper we investigate the existence of minimizers to a recently developped isotropic Cosserat shell model [6, 21], including higher order terms. The Cosserat shell model naturally includes an independent triad of rigid directors, which are coupled to the shell-deformation. From an engineering point of view, such models are preferred, since the independent rotation field allows for transparent coupling between shell and beam parts. It is interesting that the kinematical structure of 6-parameter shells [19, 41, 5] (involving the translation vector and rotation tensor) is identical to the kinematical structure of Cosserat shells (defined as material surfaces endowed with a triad of rigid directors describing the orientation of points). Using the derivation approach, Neff [28, 33, 51, 30, 31] has modelled and analysed the so-called nonlinear planar-Cosserat shell models, in which a full triad of orthogonal directors, independent of the normal of the shell, is taken into account. The results have been obtained by an 8-parameter ansatz of the deformation through the thickness and consistent analytic integration over the thickness in the case of a flat undeformed shell reference configuration. In previous papers, we have extended the modelling from flat shells to the most general case of initially curved shells [6, 21]. Our ansatz allows for a consistent shell model up to order $O(h^5)$ in the shell thickness. Interestingly, all $O(h^5)$-terms in the shell energy depend on the initial curvature of the shell and vanish for a flat shell. However, all occurring material coefficients of the shell model are uniquely determined in terms of the isotropic underlying three-dimensional Cosserat bulk-model and the given initial geometry of the shell. Thus, we fill a certain gap in the general 6-parameter shell theory, since all known hitherto models leave the precise structure of the constitutive equations wide open. In the present paper, we will show that our model is mathematically well-posed in the sense that global minimizers exist.

The topic of existence of solutions for the 2D equations of linear and nonlinear elastic shells has been treated in many works. The results that can be found in the literature refer to various types of shell models and they employ different techniques, see e.g. [24, 25, 45, 20, 47, 48, 44, 46, 4, 23, 2]. The existence theory for linear or nonlinear shells is presented in details in the books of Ciarlet [12, 13, 14], together with many historical remarks and bibliographic references. A fruitful approach to the existence theory of 2D plate and shell models (obtained as limit cases of 3D models) is the Γ-convergence analysis of thin structures, see e.g. [35, 32, 36, 40]. By ignoring the Cosserat effects, in order to start with a well-posed three dimensional model, it is mandatory to consider a polyconvex energy [3] in the three-dimensional formulation of the initial problem. In this direction, an example is the article [17], see also [16, 9], where the Ciarlet-Geymonat energy [15] is used. In these articles, no through the thickness integration is performed analytically and no reduced completely two-dimensional minimization problem is presented. The obtained problems are "two-dimensional" only in the sense that the final problem is to find three vector fields on a bounded open subset of $\mathbb{R}^2$, but all three-dimensional coordinates remain present in the minimization problem. By contrast, when a nonlinear three-dimensional problem in the Cosserat theory is considered, the three-dimensional problem is well-posed [34, 50, 49] and permits a complete dimensional reduction.

The classical geometrically nonlinear Kirchhoff-Love model (the Koiter model for short), is given by the minimization problem with respect to the midsurface deformation $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ of the type

$$\begin{aligned}\int_\omega \Big\{ h\Big(\mu\|(\mathrm{I}_m - \mathrm{I}_{y_0})\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\mathrm{tr}\Big[(\mathrm{I}_m - \mathrm{I}_{y_0})\Big]^2\Big)\\ +\frac{h^3}{12}\Big(\mu\|(\mathrm{II}_m - \mathrm{II}_{y_0})\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\mathrm{tr}\Big[(\mathrm{II}_m - \mathrm{II}_{y_0})\Big]^2\Big)\Big\}\,\mathrm{d}a,\end{aligned} \tag{1.1}$$

where $\mathrm{I}_m := [\nabla m]^T\,\nabla m \in \mathbb{R}^{2\times 2}$ and $\mathrm{II}_m := -[\nabla m]^T\,\nabla m \in \mathbb{R}^{2\times 2}$ are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form* on $m(\omega)$, respectively. However, this problem is notoriously ill-posed, since the first membrane term is non-convex in $\nabla\, m$ and indeed a non-rank-one elliptic expression. Even the inclusion of the bending terms is not sufficient to regularize the problem [28, 33]. The very same problem arises in geometrically nonlinear Reissner-Mindlin (Naghdi) type shell models, which already include an independent director-vector-field that does not coincide with the normal to the surface, as in the Kirchhoff-Love model.

Let us explain the typical situation by looking at representative energy terms for the different models. Assume that $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is the deformation of the midsurface of a flat shell, n_m is the unit normal to the shell midsurface, the unit vector $d : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is an independent director vector-field, and $\overline{R}^T : \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$

is an independent rotation field. Then the essence of a Kirchhoff-Love planar shell model is represented by the minimization problem with respect to $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ of the type

$$\text{“Kirchhoff-Love type”} \qquad \int_\omega \Big\{ h \underbrace{\|(\nabla m \,|\, n_m)^T (\nabla m \,|\, n_m) - 1\!\!1_3\|^2}_{\text{“membrane”}} + \frac{h^3}{12} \underbrace{\|\nabla n_m\|^2}_{\text{“bending”}} \Big\} \, \mathrm{d}a \tag{1.2}$$
$$= \int_\omega \Big\{ h \, \|(\nabla m)^T (\nabla m) - 1\!\!1_2\|^2 + \frac{h^3}{12} \|\nabla n_m\|^2 \Big\} \, \mathrm{d}a.$$

The essence of the corresponding Reissner-Mindlin problem is represented by the minimization problem with respect to (m, d) of the type

$$\text{“Reissner-Mindlin type”} \qquad \int_\omega \Big\{ h \, \|(\nabla m \,|\, d)^T (\nabla m \,|\, d) - 1\!\!1_3\|^2 + \frac{h^3}{12} \|\nabla d\|^2 \Big\} \, \mathrm{d}a. \tag{1.3}$$

And finally, the Cosserat flat shell model has the structure given by the minimization problem with respect to $(m, \overline{R})$ of the type

$$\text{“Cosserat-shell”} \qquad \int_\omega \Big\{ h \, \|\overline{R}^T (\nabla m \,|\, \overline{R}.e_3) - 1\!\!1_3\|^2 + \frac{h^3}{12} \|\nabla \overline{R}\|^2 \Big\} \, \mathrm{d}a. \tag{1.4}$$

Problems (1.2) and (1.3) are non-elliptic with respect to m at given d, while problem (1.4) is even linear with respect to m at given rotation field $\overline{R}$, which is itself controlled by the curvature term $\|\nabla \overline{R}\|^2$. Therefore, in principle, (1.4) admits minimizers, while (1.2) and (1.3) in general do not.

In view of these mathematical deficiencies, in the literature we find many types of existence theorems, which treat certain approximations of (1.1). The above mentioned approach by Ciarlet and his co-authors [17, 16, 9] falls into this category. It has already been noted by Neff [28], that an independent control of the continuum rotations in quadratic, non-rank-one convex energies like the membrane-term in (1.1) is sufficient to resolve the non-rank-one convexity issue. This is precisely, what the Cosserat shell model is incorporating from the outset by considering not a single director as additional independent field, but a triad of rigid directors - the rotation field $\overline{R} \in \mathrm{SO}(3)$.

Concerning the geometrically nonlinear theory of elastic Cosserat shells with drilling rotations including $O(h^3)$-terms, there is no existence theorem published in the literature, except [7], as far as we are aware of. Existence results for the related Cosserat model of initially planar shells have been obtained earlier by Neff [28, 33]. For our new model, we search for the minimizing solution pair of class $\mathrm{H}^1(\omega, \mathbb{R}^3)$ for the translation vector and $\mathrm{H}^1(\omega, \mathrm{SO}(3))$ for the rotation tensor. For the proof of existence, we employ the direct methods of the calculus of variations, extensions of the techniques presented in [28, 33, 7, 8], coercivity and uniform convexity of the energy in the appropriate geometrically nonlinear strain and curvature measure. A first task is to show the existence of the solution for the theory including $O(h^5)$-terms. In this case the expression of the energy allows us to have a decent control on each term of the energy density, in order to show the coercivity and the convexity of the energy. A second task is to consider that part of the energy which contains only $O(h^3)$-terms. In this case the obtained minimization problem is not the same as that considered in [18, 10, 11, 19, 7, 8], since additional bending-curvature and curvature energy-terms are included and the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem. For the $O(h^3)$-model, the problem of coercivity turns out to be more involved, since some steps used to prove the coercivity for the $O(h^5)$-model cannot be done in the same manner. As a preparation for the existence proofs we will rewrite the energy in an equivalent form that allows us to prove the coercivity and convexity of the energy. Moreover, for the $O(h^3)$-model, we need to impose either a stronger assumption on the constitutive parameters or a relation between the thickness and the internal length. This behaviour highlights the importance and interest of including $O(h^5)$-terms.

2 The new geometrically nonlinear Cosserat shell model

2.1 Notation

In this paper, for $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated (squared) vector norm $\|a\|^2_{\mathbb{R}^n} = \langle a, a \rangle_{\mathbb{R}^n}$ The standard Euclidean scalar product on the set of real $n \times m$ second order tensors $\mathbb{R}^{n\times m}$ is given by $\langle X, Y\rangle_{\mathbb{R}^{n\times m}} = \mathrm{tr}(X\,Y^T)$, and thus the (squared) Frobenius tensor norm is $\|X\|^2_{\mathbb{R}^{n\times m}} = \langle X, X\rangle_{\mathbb{R}^{n\times m}}$. In the following we omit the subscripts $\mathbb{R}^n, \mathbb{R}^{n\times m}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}_n$, so that $\mathrm{tr}(X) = \langle X, \mathbb{1}_n\rangle$. We let $\mathrm{Sym}(n)$ and $\mathrm{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-group theory, e.g., $\mathrm{GL}(n) = \{X \in \mathbb{R}^{n\times n}\,|\det(X) \neq 0\}$ the general linear group, $\mathrm{SO}(n) = \{X \in \mathrm{GL}(n)|X^T X = \mathbb{1}_n, \det(X) = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n}\,|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(n) = \{X \in \mathbb{R}^{n\times n}\,|\,\mathrm{tr}(X) = 0\}$ of traceless tensors. For all $X \in \mathbb{R}^{n\times n}$ we set $\mathrm{sym}\,X = \frac{1}{2}(X + X^T) \in \mathrm{Sym}(n)$, $\mathrm{skew}\,X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(n)$ and the deviatoric part $\mathrm{dev}\,X = X - \frac{1}{n}\,\mathrm{tr}(X)\,\mathbb{1}_n \in \mathfrak{sl}(n)$ and we have the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(n) = \{\mathfrak{sl}(n) \cap \mathrm{Sym}(n)\} \oplus \mathfrak{so}(n) \oplus \mathbb{R}\cdot\mathbb{1}_n$, $X = \mathrm{dev}\,\mathrm{sym}\,X + \mathrm{skew}\,X + \frac{1}{n}\,\mathrm{tr}(X)\,\mathbb{1}_n$. A matrix having the three columns vectors A_1, A_2, A_3 will be written as $(A_1\,|\,A_2\,|\,A_3)$. We make use of the operator $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ associating with a matrix $A \in \mathfrak{so}(3)$ the vector $\mathrm{axl}\,A := (-A_{23}, A_{13}, -A_{12})^T$. The inverse of the operator $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ is denoted by $\mathrm{anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$.

Let Ω be an open domain of $\mathbb{R}^3$. The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$, $\mathbb{R}^{3\times 3}$ or $\mathrm{SO}(3)$, respectively will be denoted by $\mathrm{L}^2(\Omega;\mathbb{R})$, $\mathrm{L}^2(\Omega;\mathbb{R}^3)$, $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3})$ and $\mathrm{L}^2(\Omega;\mathrm{SO}(3))$, respectively. Moreover, we use the standard Sobolev spaces $\mathrm{H}^1(\Omega;\mathbb{R})$ [1, 22, 26] of functions u. For vector fields $u = (u_1, u_2, u_3)^T$ with $u_i \in \mathrm{H}^1(\Omega)$, $i = 1, 2, 3$, we define $\nabla\,u := (\nabla\,u_1\,|\,\nabla\,u_2\,|\,\nabla\,u_3)^T$. The corresponding Sobolev-space will be denoted by $\mathrm{H}^1(\Omega;\mathbb{R}^3)$. If a tensor $Q : \Omega \to \mathrm{SO}(3)$ has the components in $\mathrm{H}^1(\Omega;\mathbb{R})$, then we mark this by writing $Q \in \mathrm{H}^1(\Omega;\mathrm{SO}(3))$. When writting the norm in the corresponding Sobolev-space we will specify the space in subscript. The space will be omitted only when the Frobenius norm or scalar product is considered.

2.2 The deformation of Cosserat shells

Let $\Omega_\xi \subset \mathbb{R}^3$ be a three-dimensional *shell-like thin domain*. In a fixed standard base e_1, e_2, e_3 of $\mathbb{R}^3$, a generic point of Ω_ξ will be denoted by (ξ_1, ξ_2, ξ_3). The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration Ω_ξ is assumed to be a natural state. The deformation of the body occupying the domain Ω_ξ is described by a vector map $\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathbb{R}^3$ (*called deformation*) and by a *microrotation* tensor $\overline{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathrm{SO}(3)$. We denote the current configuration (deformed configuration) by $\Omega_c := \varphi_\xi(\Omega_\xi) \subset \mathbb{R}^3$, see Figure 1.

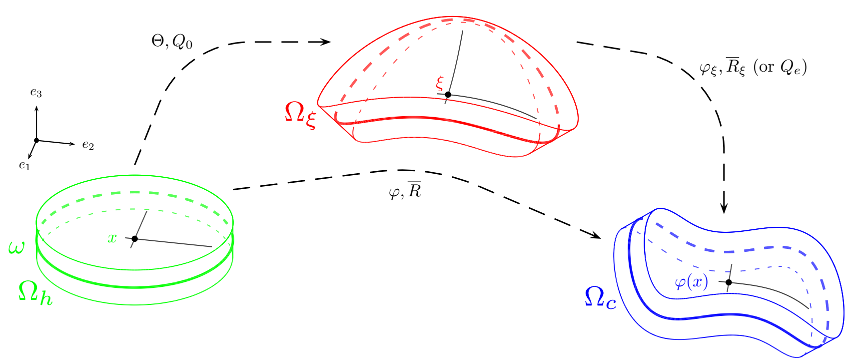

Figure 1: The shell in its initial configuration Ω_ξ, the shell in the deformed configuration Ω_c, and the fictitious planar Cartesian reference configuration Ω_h. Here, $\overline{R}_\xi$ is the elastic rotation field, Q_0 is the initial rotation from the fictitious planar Cartesian reference configuration to the initial configuration Ω_ξ, and $\overline{R}$ is the total rotation field from the fictitious planar Cartesian reference configuration to the deformed configuration Ω_c.

In what follows, we consider the *fictitious Cartesian (planar) configuration* Ω_h of the body. This parameter domain $\Omega_h \subset \mathbb{R}^3$ is a right cylinder of the form

$$\Omega_h = \left\{(x_1, x_2, x_3)\;\Big|\;(x_1, x_2) \in \omega,\ -\frac{h}{2} < x_3 < \frac{h}{2}\right\} = \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),$$

where $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\omega$ and the constant length $h > 0$ is the *thickness of the shell*. For shell–like bodies we consider the domain Ω_h to be thin, i.e. the thickness h is small. We assume furthermore that there exists a C^1-diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ in the specific form

$$\Theta(x_1, x_2, x_3) \ = \ y_0(x_1, x_2) + x_3 \ n_0(x_1, x_2), \qquad n_0 \ = \ \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}\,, \tag{2.1}$$

where $y_0 : \omega \to \mathbb{R}^3$ is a function of class $C^2(\omega)$, so that Θ maps the fictitious planar Cartesian parameter space Ω_h onto the initially curved reference configuration of the shell $\Theta(\Omega_h) = \Omega_\xi$, $\Theta(x_1, x_2, x_3) = (\xi_1, \xi_2, \xi_3)$. The diffeomorphism Θ maps the midsurface ω of the fictitious Cartesian parameter space Ω_h onto the midsurface $\omega_\xi = y_0(\omega)$ of Ω_ξ and n_0 is the unit normal vector to ω_ξ. For simplicity and where no confusions may arise, we will omit subsequently to write explicitly the arguments (x_1, x_2, x_3) of the diffeomorphism Θ or we will specify only its dependence on x_3. We use the polar decomposition [37] of $\nabla_x\Theta(x_3)$ and write $\nabla_x\Theta(x_3) = Q_0(x_3)\, U_0(x_3)\,$, $Q_0(x_3) = \mathrm{polar}(\nabla_x\Theta)(x_3) \in \mathrm{SO}(3)$, $U_0(x_3) \in \mathrm{Sym}^+(3)$. Let us remark that

$$\nabla_x\Theta(x_3) \ = \ (\nabla y_0|n_0) + x_3(\nabla n_0|0) \ \ \forall\, x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \ \ \nabla_x\Theta(0) \ = \ (\nabla y_0|\, n_0), \ \ [\nabla_x\Theta(0)]^{-T}\!.\, e_3 \ = n_0, \tag{2.2}$$

and that $\det(\nabla y_0|n_0) = \sqrt{\det[(\nabla y_0)^T\nabla y_0]}$ represents the surface element.

In the following, we consider the *Weingarten map*[1] *(or shape operator)* on $y_0(\omega)$ defined by its associated matrix $\mathrm{L}_{y_0} \ = \ \mathrm{I}_{y_0}^{-1}\mathrm{II}_{y_0} \in \mathbb{R}^{2\times 2}$, where $\mathrm{I}_{y_0} := [\nabla y_0]^T\,\nabla y_0 \in \mathbb{R}^{2\times 2}$ and $\mathrm{II}_{y_0} \ := \ -[\nabla y_0]^T\,\nabla n_0 \in \mathbb{R}^{2\times 2}$ are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form*, respectively. Then, the *Gauß curvature* K of the surface $y_0(\omega)$ is determined by $\mathrm{K} := \det(\mathrm{L}_{y_0})$ and the *mean curvature* H through $2\,\mathrm{H} := \mathrm{tr}(\mathrm{L}_{y_0})$. We also need the tensors defined by:

$$\mathrm{A}_{y_0} := (\nabla y_0|0)\ [\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3}, \qquad \mathrm{B}_{y_0} := -(\nabla n_0|0)\ [\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3}, \tag{2.3}$$

and the so-called *alternator tensor* C_{y_0} of the surface [52]

$$\mathrm{C}_{y_0} := \det(\nabla_x\Theta(0))\,[\nabla_x\Theta(0)]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x\Theta(0)]^{-1}. \tag{2.4}$$

Now, let us define the map $\varphi : \Omega_h \to \Omega_c,\ \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3))$. We view φ as a function which maps the fictitious planar reference configuration Ω_h into the deformed configuration Ω_c. We also consider the *elastic microrotation* $\overline{Q}_{e,s} : \Omega_h \to \mathrm{SO}(3),\ \overline{Q}_{e,s}(x_1, x_2, x_3) := \overline{R}_\xi(\Theta(x_1, x_2, x_3))\,$.

In [21], by assuming that $\overline{Q}_{e,s}(x_1, x_2, x_3) = \overline{Q}_{e,s}(x_1, x_2)$ and considering an *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like body, we have obtained a two-dimensional minimization problem in which the energy density is expressed in terms of the following tensor fields on the surface ω

$$\begin{aligned} \mathcal{E}_{m,s} &:= \ \overline{Q}_{e,s}^T(\nabla m|\overline{Q}_{e,s}\nabla_x\Theta(0).e_3)[\nabla_x\Theta(0)]^{-1} - \mathbb{1}_3 && \text{(elastic shell strain tensor)}, \qquad (2.5)\\ \mathcal{K}_{e,s} &:= \ (\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)[\nabla_x\Theta(0)]^{-1} && \text{(elastic shell bending–curvature tensor)}, \end{aligned}$$

where $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ represents the deformation of the midsurface. When these measures vanish, the shell undergoes a rigid body motion. Indeed, $\mathcal{K}_{e,s}= 0$ implies $\partial_{x_1}\overline{Q}_{e,s} = 0$, $\partial_{x_2}\overline{Q}_{e,s} = 0$, while $\mathcal{E}_{m,s} = 0$ leads to $\nabla m = \overline{Q}_{e,s}\nabla y_0$. Since $\overline{Q}_{e,s}$ is constant and $m = \overline{Q}_{e,s}\, y_0 + c$, where c is a constant vector field, this means that the shell is in a rigid body motion with constants translation c and constant rotation $\overline{Q}_{e,s}$.

2.3 Formulation of the minimization problem

In [21], we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \to \mathrm{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \overline{Q}_{e,s})$ the functional

$$I(m, \overline{Q}_{e,s}) = \int_\omega \Big[W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) + W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big)\Big] \det(\nabla y_0|n_0)\, \mathrm{d}a - \overline{\Pi}(m, \overline{Q}_{e,s})\,, \tag{2.6}$$

[1] We identify the Weingarten map, the first fundamental form and the second fundamental form with their associated matrices in the fixed base vector e_1, e_2, e_3.

where the membrane part $W_{\rm memb}\big(\mathcal{E}_{m,s}\big)$, the membrane–bending part $W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\,\mathcal{K}_{e,s}\big)$ and the bending–curvature part $W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)$ of the shell energy density are given by

$$\begin{aligned}
W_{\rm memb}\big(\mathcal{E}_{m,s}\big) &= \Big(h+{\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big),\\
W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\,\mathcal{K}_{e,s}\big) &= \Big(\frac{h^3}{12}-{\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\,{\rm B}_{y_0}+{\rm C}_{y_0}\mathcal{K}_{e,s}\big) \qquad (2.7)\\
&\quad-\frac{h^3}{3}{\rm H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\mathcal{E}_{m,s}{\rm B}_{y_0}+{\rm C}_{y_0}\,\mathcal{K}_{e,s}\big)+\frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},(\mathcal{E}_{m,s}{\rm B}_{y_0}+{\rm C}_{y_0}\,\mathcal{K}_{e,s}){\rm B}_{y_0}\big)\\
&\quad+\frac{h^5}{80}\,W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0}+{\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\,\big),\\
W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) &= \Big(h-{\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)+\Big(\frac{h^3}{12}-{\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\,\big)+\frac{h^5}{80}\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}^2\big),
\end{aligned}$$

with

$$\begin{aligned}
W_{\rm shell}(S) &= \mu\,\|{\rm sym}\,S\|^2+\mu_{\rm c}\|{\rm skew}\,S\|^2+\frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[{\rm tr}(S)\big]^2, \qquad (2.8)\\
\mathcal{W}_{\rm shell}(S,T) &= \mu\,\big\langle{\rm sym}\,S,\,{\rm sym}\,T\big\rangle+\mu_{\rm c}\big\langle{\rm skew}\,S,\,{\rm skew}\,T\big\rangle+\frac{\lambda\,\mu}{\lambda+2\,\mu}\,{\rm tr}(S)\,{\rm tr}(T),\\
W_{\rm mp}(S) &= \mu\,\|{\rm sym}\,S\|^2+\mu_{\rm c}\|{\rm skew}\,S\|^2+\frac{\lambda}{2}\,\big[{\rm tr}(S)\,\big]^2=\mathcal{W}_{\rm shell}(S)+\frac{\lambda^2}{2\,(\lambda+2\,\mu)}\,[{\rm tr}(S)]^2,\\
W_{\rm curv}(S) &= \mu\,L_{\rm c}^2\,\big(b_1\,\|{\rm dev\,sym}\,S\|^2+b_2\,\|{\rm skew}\,S\|^2+b_3\,[{\rm tr}(S)]^2\big)\,.
\end{aligned}$$

The parameters μ and λ are the *Lamé constants* of classical isotropic elasticity, $\kappa=\frac{2\,\mu+3\,\lambda}{3}$ is the *infinitesimal bulk modulus*, b_1,b_2,b_3 are *non-dimensional constitutive curvature coefficients (weights)*, $\mu_{\rm c}\geq 0$ is called the *Cosserat couple modulus* and ${\rm L}_{\rm c}>0$ introduces an *internal length* which is characteristic for the material, e.g. related to the grain size in a polycrystal. The internal length $L_{\rm c}>0$ is responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that $\mu>0$, $\kappa>0$, $\mu_{\rm c}>0$, $b_1>0$, $b_2>0$, $b_3>0$. Here, $\mu\,L_{\rm c}^2$ plays the role of a *dimensional agreement factor*. Without loss of generality, we may assume that $0<b_1<1$, $0<b_2<1$, $0<b_3<1$. All constitutive coefficients are deduced from the three-dimensional formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads $\overline{\Pi}(m,\overline{Q}_{e,s})$ appearing in (2.6) is expressed by

$$\begin{aligned}
\overline{\Pi}(m,\overline{Q}_{e,s}) &= \Pi_\omega(m,\overline{Q}_{e,s})+\Pi_{\gamma_t}(m,\overline{Q}_{e,s})\,,\qquad\text{with} \qquad (2.9)\\
\Pi_\omega(m,\overline{Q}_{e,s}) &= \int_\omega\big\langle f,u\big\rangle\,{\rm da}+\Lambda_\omega(\overline{Q}_{e,s})\qquad\text{and}\qquad \Pi_{\gamma_t}(m,\overline{Q}_{e,s})=\int_{\gamma_t}\big\langle t,u\big\rangle\,ds+\Lambda_{\gamma_t}(\overline{Q}_{e,s})\,,
\end{aligned}$$

where $u(x_1,x_2)=m(x_1,x_2)-y_0(x_1,x_2)$ is the displacement vector of the midsurface, $\Pi_\omega(m,\overline{Q}_{e,s})$ is the potential of the external surface loads f, while $\Pi_{\gamma_t}(m,\overline{Q}_{e,s})$ is the potential of the external boundary loads t. Here, γ_t and γ_d are nonempty subsets of the boundary of ω such that $\gamma_t\cup\gamma_d=\partial\omega$ and $\gamma_t\cap\gamma_d=\emptyset$. On γ_t we have considered traction boundary conditions, while on γ_d we have the Dirichlet-type boundary conditions:

$$m\big|_{\gamma_d}=m^*,\qquad\text{simply supported (fixed, welded)},\qquad \overline{Q}_{e,s}\big|_{\gamma_d}=\overline{Q}^*_{e,s},\qquad\text{clamped}^2, \qquad (2.10)$$

where the boundary conditions are to be understood in the sense of traces.

The functions $\Lambda_\omega\,,\Lambda_{\gamma_t}:{\rm L}^2(\omega,{\rm SO}(3))\to\mathbb{R}$ are expressed in terms of the loads from the three-dimensional parental variational problem, see [21], and they are assumed to be continuous and bounded operators.

Remark 2.1. *Our model [21] is constructed under the following assumptions upon the thickness*

$$h\,|\kappa_1|<\frac{1}{2}\qquad\textit{and}\qquad h\,|\kappa_2|<\frac{1}{2},$$

where κ_1 and κ_2 denote the principal curvatures of the surface.

[2] The existence theory works also for free microrotations at the boundary since SO(3) is a compact manifold.

We will consider materials for which the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ and Young's modulus $E = \frac{\mu(3\,\lambda+2\,\mu)}{\lambda+\mu}$ are such that $-\frac{1}{2} < \nu < \frac{1}{2}$ and $E > 0$. This assumption implies that $2\,\lambda + \mu > 0$. Under these assumptions on the constitutive coefficients, together with the positivity of μ, μ_c, b_1, b_2 and b_3, and the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3)$, since

$$W_{\rm shell}(S) \;=\; \mu\, \|{\rm dev}\,{\rm sym}\, S\|^2 + \mu_{\rm c} \|{\rm skew}\, S\|^2 + \frac{2\,\mu\,(2\,\lambda+\mu)}{3(\lambda+2\,\mu)}\,[{\rm tr}(S)]^2, \tag{2.11}$$

it follows that there exists the positive constants c_1^+, c_2^+, C_1^+ and C_2^+ such that

$$C_1^+\|S\|^2 \geq\; W_{\rm shell}(S) \geq\; c_1^+\|S\|^2, \qquad\quad C_2^+\|S\|^2 \geq\; W_{\rm curv}(S) \geq\; c_2^+\|S\|^2 \qquad \forall\, S\in \mathbb{R}^{3\times 3}. \tag{2.12}$$

Hence, we note

$$W_{\rm mp}(S) = W_{\rm shell}(S) + \frac{\lambda^2}{2(\lambda+2\,\mu)}\,({\rm tr}(S))^2 \geq\; W_{\rm shell}(S) \geq\; c_1^+\|S\|^2. \tag{2.13}$$

3 Existence of minimizers for the Cosserat shell model of order $O(h^5)$

In order to establish an existence result by the direct methods of the calculus of variations, we need to show the coercivity of the elastically stored shell energy density.

3.1 Coercivity and uniform convexity in the theory of order $O(h^5)$

Proposition 3.1. [Coercivity in the theory including terms up to order $O(h^5)$] *For sufficiently small values of the thickness h such that $h|\kappa_1| < \frac{1}{2}$ and $h|\kappa_2| < \frac{1}{2}$ and for constitutive coefficients satisfying $\mu > 0$, $\mu_c > 0$, $2\,\lambda+\mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$, the energy density*

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) = W_{\rm memb}\big(\mathcal{E}_{m,s}\big) + W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\,\mathcal{K}_{e,s}\big) + W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) \tag{3.1}$$

is coercive in the sense that there exists a constant $a_1^+ > 0$ such that

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \;\geq\; a_1^+\,\big(\|\mathcal{E}_{m,s}\|^2 + \|\mathcal{K}_{e,s}\|^2\,\big), \tag{3.2}$$

where a_1^+ depends on the constitutive coefficients.

Proof. In order to prove the coercivity note that the principal curvatures κ_1, κ_2 are the solutions of the characteristic equation of ${\rm L}_{y_0}$, i.e. $\kappa^2 - {\rm tr}({\rm L}_{y_0})\,\kappa + \det({\rm L}_{y_0}) \;=\; \kappa^2 - 2\,{\rm H}\,\kappa + {\rm K} \;=\; 0$. Therefore, from the assumptions $h\,|\kappa_1| < \frac{1}{2},\;\; h\,|\kappa_2| < \frac{1}{2}$, it follows that

$$h^2|K| = h^2\,|\kappa_1|\,|\kappa_2| < \frac{1}{4} \qquad \text{and} \qquad 2\,h\,|H| = h\,|\kappa_1+\kappa_2| < 1. \tag{3.3}$$

Therefore, $h - {\rm K}\,\frac{h^3}{12} > 0$ and $\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80} > 0$ and

$$\begin{aligned} W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq{}& \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big) + \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}\big) \\ &- \frac{h^3}{3}\,|{\rm H}|\;|\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}\big)| - \frac{h^3}{12}\;2\,|\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},(\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}){\rm B}_{y_0}\big)| \\ &+ \frac{h^5}{80}\;W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\,\big) + \Big(h - {\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big). \end{aligned} \tag{3.4}$$

Using the Cauchy–Schwarz inequality we deduce

$$\begin{aligned} W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq{}& \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{1}{3}\,|{\rm H}|\;\big[h^2\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)\big]^{\frac{1}{2}}\;\big[h^4\,W_{\rm shell}\big(\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}\big)\big]^{\frac{1}{2}} \\ &+ \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}\big) \\ &- \frac{1}{6}\,\Big[h\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)\Big]^{\frac{1}{2}}\,\Big[h^5 W_{\rm shell}\big((\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}){\rm B}_{y_0}\big)\Big]^{\frac{1}{2}} \\ &+ \frac{h^5}{80}\;W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\,\big) + \Big(h - {\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big). \end{aligned} \tag{3.5}$$

The arithmetic-geometric mean inequality leads to the estimate

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})\geq\Big(h+\mathrm{K}\,\frac{h^3}{12}-\frac{h^2}{6}\varepsilon\,|\mathrm{H}|\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)+\Big(\frac{h^3}{12}-\mathrm{K}\,\frac{h^5}{80}-\frac{h^4}{6\,\varepsilon}\,|\mathrm{H}|\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)$$
$$-\,\frac{h}{12}\delta\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)-\frac{h^5}{12\,\delta}W_{\rm shell}\big((\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big) \tag{3.6}$$
$$+\,\frac{h^5}{80}\,W_{\rm mp}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\,\big)+\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)\quad\forall\,\varepsilon>0\text{ and }\delta>0.$$

Using (2.13), we obtain

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})\geq\Big(h-\frac{h}{12}\delta+\mathrm{K}\,\frac{h^3}{12}-\frac{h^2}{6}\varepsilon\,|\mathrm{H}|\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)$$
$$+\,\Big(\frac{h^3}{12}-\mathrm{K}\,\frac{h^5}{80}-\frac{h^4}{6\,\varepsilon}\,|\mathrm{H}|\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)$$
$$+\,\Big(\frac{h^5}{80}-\frac{h^5}{12\,\delta}\Big)\,W_{\rm shell}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\,\big) \tag{3.7}$$
$$+\,\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)\qquad\forall\,\varepsilon>0\quad\text{and}\quad\delta>0.$$

Taking $\delta=8$ and and $\varepsilon=2$ we get[3] that

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})\geq h\,\Big[\frac{1}{3}-\mathrm{K}\,\frac{h^2}{12}-\frac{h}{3}\,|\mathrm{H}|\Big]\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)+\frac{h^3}{12}\Big(1-|\mathrm{K}|\,\frac{12\,h^2}{80}-h\,|\mathrm{H}|\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)$$
$$+\,\Big(h-|\mathrm{K}|\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big). \tag{3.8}$$

In view of (3.3) and (2.12), we deduce

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})\geq h\,\frac{7}{48}\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)+\frac{h^3}{12}\,\frac{37}{80}\,W_{\rm shell}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)+h\,\frac{47}{48}\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)$$
$$\geq h\,\frac{7}{48}\,c_1^+\,\|\mathcal{E}_{m,s}\|^2+\frac{h^3}{12}\,\frac{37}{80}\,c_1^+\,\|\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2+h\,\frac{47}{48}\,c_2^+\|\mathcal{K}_{e,s}\|^2. \tag{3.9}$$

The desired constant a_1^+ from the conclusion can be chosen as $a_1^+=\min\big\{h\,\frac{7}{48}\,c_1^+,h\,\frac{47}{48}\,c_2^+\big\}$. ■

Corollary 3.2. [Uniform convexity in the theory including terms up to order $O(h^5)$] *For sufficiently small values of the thickness h such that $h|\kappa_1|<\frac{1}{2}$ and $h|\kappa_2|<\frac{1}{2}$ and for constitutive coefficients such that $\mu>0$, $\mu_{\rm c}>0$, $2\,\lambda+\mu>0$, $b_1>0$, $b_2>0$ and $b_3>0$, the energy density*

$$W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})=W_{\rm memb}\big(\mathcal{E}_{m,s}\big)+W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\,\mathcal{K}_{e,s}\big)+W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) \tag{3.10}$$

is uniformly convex in $(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$, i.e. there exists a constant $a_1^+>0$ such that

$$D^2\,W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,[(H_1,H_2),(H_1,H_2)]\geq a_1^+(\|H_1\|^2+\|H_2\|^2)\qquad\forall\,H_1,H_2\in\mathbb{R}^{3\times3}. \tag{3.11}$$

Proof. For a bilinear expression $W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ in terms of $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$, the second derivative with respect to these argument variables coincides with the function itself, modulo a scalar multiplication. We will prove this known fact only for two terms of the energy and we show that

$$D^2(\|{\rm sym}\,\mathcal{E}_{m,s}\|^2).\,[(H_1,H_2),(H_1,H_2)]=2\,\|{\rm sym}\,H_1\|^2\qquad\text{and} \tag{3.12}$$
$$D^2(\big\langle{\rm sym}\,\mathcal{E}_{m,s},{\rm sym}(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\big\rangle).\,[(H_1,H_2),(H_1,H_2)]=2\,\big\langle{\rm sym}\,H_1,{\rm sym}(H_1\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,H_2)\big\rangle.$$

[3] This step cannot be repeated in the proof of the coercivity up to order $O(h^3)$, since $W_{\rm shell}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\,\big)$ cannot be skipped. This is the reason why we have to choose another strategy to obtain the desired estimates.

Indeed, on the one hand, we have $D_F(\|F\|^2).\, H = 2\,\langle F, H\,\rangle$, $\big\langle D^2_F(\|F\|^2).\, H, H\big\rangle = 2\,\|H\|^2$. Useful in our calculation is that

$$\begin{aligned} D^2\, W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,[(H_1,H_2),(H_1,H_2)] = & D^2_{\mathcal{E}_{m,s},\mathcal{E}_{m,s}}\, W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,(H_1,H_1) \\ & + 2\, D_{\mathcal{K}_{e,s}}[D_{\mathcal{E}_{m,s}}\, W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).(H_1,H_1)].(H_2,H_2) \\ & + D^2_{\mathcal{K}_{e,s},\mathcal{K}_{e,s}}\, W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,(H_2,H_2). \end{aligned}$$

Since $\mathrm{sym}:\mathbb{R}^{3\times 3}\to \mathrm{Sym}(3)$ is a linear operator, we obtain

$$D^2(\|\mathrm{sym}\,\mathcal{E}_{m,s}\|^2).\,[(H_1,H_2),(H_1,H_2)] = D^2_{\mathcal{E}_{m,s},\mathcal{E}_{m,s}}\,(\|\mathrm{sym}\,\mathcal{E}_{m,s}\|^2).\,(H_1,H_1) = 2\,\|\mathrm{sym}\,H_1\|^2, \tag{3.13}$$

which proves $(3.12)_1$. On the other hand, it holds

$$\begin{aligned} D^2_{\mathcal{E}_{m,s},\mathcal{E}_{m,s}}(\big\langle \mathrm{sym}\,\mathcal{E}_{m,s}, \mathrm{sym}(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\big\rangle).\,(H_1,H_1) &= 2\,\big\langle \mathrm{sym}\,H_1, H_1\mathrm{B}_{y_0}\big\rangle, \\ D^2_{\mathcal{K}_{e,s},\mathcal{K}_{e,s}}(\big\langle \mathrm{sym}\,\mathcal{E}_{m,s}, \mathrm{sym}(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\big\rangle).\,(H_2,H_2) &= 0, \\ D_{\mathcal{K}_{e,s}}[D_{\mathcal{E}_{m,s}}\,(\big\langle \mathrm{sym}\,\mathcal{E}_{m,s}, \mathrm{sym}(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\big\rangle).(H_1,H_1)].(H_2,H_2) &= \big\langle \mathrm{sym}\,H_1, \mathrm{C}_{y_0}\,H_2\big\rangle. \end{aligned} \tag{3.14}$$

Therefore

$$D^2(\big\langle \mathrm{sym}\,\mathcal{E}_{m,s}, \mathrm{sym}(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\big\rangle).\,[(H_1,H_2),(H_1,H_2)] = 2\,\big\langle \mathrm{sym}\,H_1, \mathrm{sym}(H_1\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,H_2)\big\rangle, \tag{3.15}$$

which proves $(3.12)_2$. In conclusion, after making similar calculations as above for the other terms appearing in the expression of $W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$, we obtain

$$D^2\, W(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,[(H_1,H_2),(H_1,H_2)] = 2\, W(H_1,H_2). \tag{3.16}$$

Bounding the function $W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ for all $\mathcal{E}_{m,s},\mathcal{K}_{e,s}\in\mathbb{R}^{3\times 3}$ away from zero amounts therefore to showing that $D^2W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ is positive definite. Hence, the coercivity of $W(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ expressed by Proposition 3.1 implies uniform convexity in the chosen variables. ■

3.2 The existence result in the theory of order $O(h^5)$

In this section, we prove the first main result of our paper. The admissible set $\mathcal{A}$ of solutions is defined by

$$\mathcal{A} = \big\{(m,\overline{Q}_{e,s})\in \mathrm{H}^1(\omega,\mathbb{R}^3)\times \mathrm{H}^1(\omega,\mathrm{SO}(3))\ \big|\ m\big|_{\gamma_d} = m^*,\ \overline{Q}_{e,s}\big|_{\gamma_d} = \overline{Q}^*_{e,s}\big\}, \tag{3.17}$$

where the boundary conditions are to be understood in the sense of traces.

Theorem 3.3. [Existence result for the theory including terms up to order $O(h^5)$] *Assume that the external loads satisfy the conditions*

$$f\in \mathrm{L}^2(\omega,\mathbb{R}^3), \qquad t\in \mathrm{L}^2(\gamma_t,\mathbb{R}^3), \tag{3.18}$$

and the boundary data satisfy the conditions

$$m^*\in \mathrm{H}^1(\omega,\mathbb{R}^3), \qquad \overline{Q}^*_{e,s}\in \mathrm{H}^1(\omega,\mathrm{SO}(3)). \tag{3.19}$$

Assume that the following conditions concerning the initial configuration are satisfied: $y_0:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ is a continuous injective mapping and

$$y_0\in \mathrm{H}^1(\omega,\mathbb{R}^3), \qquad Q_0(0)\in \mathrm{H}^1(\omega,\mathrm{SO}(3)), \qquad \nabla_x\Theta(0)\in \mathrm{L}^\infty(\omega,\mathbb{R}^{3\times 3}), \qquad \det[\nabla_x\Theta(0)]\geq\, a_0>0\,, \tag{3.20}$$

where a_0 is a constant. Then, for sufficiently small values of the thickness h such that $h|\kappa_1|<\frac{1}{2}$ and $h|\kappa_2|<\frac{1}{2}$ and for constitutive coefficients such that $\mu>0$, $\mu_c>0$, $2\,\lambda+\mu>0$, $b_1>0$, $b_2>0$ and $b_3>0$, the minimization problem (2.6)–(2.10) admits at least one minimizing solution pair $(m,\overline{Q}_{e,s})\in\mathcal{A}$.

Proof. We employ the direct methods of the calculus of variations, similar to [7, 34, 29]. However, in comparison to [7], due to the fact that we use only matrix notation, some steps are shortened. In Proposition 3.1 and Corollary 3.2, we have shown that the strain energy density $W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ is a quadratic convex and coercive function of $(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$.

The hypothesis (3.18) and the boundedness of Π_{S^0} and $\Pi_{\partial S^0_f}$ imply that there exists a constant $C > 0$ such that[4]

$$|\overline{\Pi}(m, \overline{Q}_{e,s})| \leq C\Big(\|m - y_0\|_{L^2(\omega)} + \|m - y_0\|_{L^2(\gamma_t)} + \|\overline{Q}_{e,s}\|_{L^2(\omega)}\Big) \quad \forall (m, \overline{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)).$$

We have $\|\overline{Q}_{e,s}\|^2 = \mathrm{tr}(\overline{Q}_{e,s}\overline{Q}_{e,s}^T) = \mathrm{tr}(\mathbb{1}_3) = 3,\ \forall \overline{Q}_{e,s} \in SO(3)$. Hence, there exists a constant $C > 0$ such that

$$|\,\overline{\Pi}(m, \overline{Q}_{e,s})\,| \leq C\,\big(\,\|m\|_{H^1(\omega)} + 1\big), \quad \forall (m, \overline{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)). \tag{3.21}$$

Considering

$$\overline{R}_s(x_1, x_2) = \overline{Q}_{e,s}(x_1, x_2)\, Q_0(x_1, x_2, 0) \in SO(3), \tag{3.22}$$

we observe that

$$\mathcal{E}_{m,s} = Q_0[\overline{R}_s^T(\nabla m|\overline{Q}_{e,s}\nabla_x\Theta(0).e_3) - Q_0^T(\nabla y_0|n_0)][\nabla_x\Theta(0)]^{-1} = Q_0(\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)[\nabla_x\Theta(0)]^{-1}. \tag{3.23}$$

The lifted quantity $\widehat{\mathrm{I}}_{y_0} = (\nabla y_0|n_0)^T(\nabla y_0|n_0) \in \mathrm{Sym}(3)$ is positive definite and also it's inverse is positive definite. Using the above relation we obtain

$$\begin{aligned}
\|\mathcal{E}_{m,s}\|^2 &= \big\langle\, Q_0^T\, Q_0(\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0), (\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)\widehat{\mathrm{I}}_{y_0}^{-1}\big\rangle \\
&= \big\langle\,\widehat{\mathrm{I}}_{y_0}^{-1}(\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)^T, (\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)^T\big\rangle \geq \lambda_0^2\,\|(\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)\|^2,
\end{aligned} \tag{3.24}$$

where λ_0 is the smallest eigenvalue of the positive definite matrix $\widehat{\mathrm{I}}_{y_0}^{-1}$. Similarly, we deduce that

$$\begin{aligned}
\|\mathcal{K}_{e,s}\|^2 &= \big\langle(\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0), (\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)\,\widehat{\mathrm{I}}_{y_0}^{-1}\big\rangle \\
&= \big\langle\widehat{\mathrm{I}}_{y_0}^{-1}(\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)^T, (\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)^T\big\rangle \\
&\geq \lambda_0^2\,\|(\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s}))\|^2.
\end{aligned} \tag{3.25}$$

From (3.24) we have

$$\|\mathcal{E}_{m,s}\|^2 \geq \lambda_0^2\Big[\|\overline{R}_s^T\nabla m\|^2 - 2\,\big\langle\overline{R}_s^T\,\nabla m,\, Q_0^T\nabla y_0\big\rangle + \|Q_0^T\nabla y_0\|^2\Big]. \tag{3.26}$$

Since $\|\overline{R}_s^T\nabla m\|^2 = \|\nabla m\|^2$ and $\|Q_0^T\nabla y_0\|^2 = \|\nabla y_0\|^2$, after integrating over ω, using (3.2), the Cauchy–Schwarz inequality and the hypothesis upon y_0, gives us the estimate

$$\begin{aligned}
\|\mathcal{E}_{m,s}\|^2_{L^2(\omega)} &\geq \lambda_0^2\Big[\|\nabla m\|^2_{L^2(\omega)} - 2\,\|\nabla m\|_{L^2(\omega)}\|\nabla y_0\|_{L^2(\omega)} + \|\nabla y_0\|^2_{L^2(\omega)}\Big] \\
&\geq \lambda_0^2\|\nabla m\|^2_{L^2(\omega)} - C_1\,\|\nabla m\|_{L^2(\omega)} + C_2,
\end{aligned} \tag{3.27}$$

for some positive constants $C_1 > 0$, $C_2 > 0$.

By virtue of the coercivity of the internal energy and (3.20), (3.21) and (3.24), the functional $I(m, \overline{Q}_{e,s})$ is bounded from below

$$\begin{aligned}
I(m, \overline{Q}_{e,s}) &\geq C_1\!\int_\omega \|\mathcal{E}_{m,s}\|^2\,\det[\nabla_x\Theta(0)]\,\mathrm{d}a - \overline{\Pi}(m, \overline{Q}_{e,s}) \geq C_2\,a_0\|\mathcal{E}_{m,s}\|^2_{L^2(\omega)} - C_3\,\big(\,\|m\|_{H^1(\omega)} + 1\big) \\
&\geq C_4\,\|\nabla m\|^2_{L^2(\omega)} - C_5\,\|m\|_{H^1(\omega)} - C_6 \quad \forall (m, \overline{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)),
\end{aligned} \tag{3.28}$$

[4]By C and C_i, $i \in \mathbb{N}$, we will denote (positive) constants that may vary from estimate to estimate but will remain independent of m, ∇m and $\overline{Q}_{e,s}$.

with $C_i > 0$, $i = 1, 2, ..., 6$. We also obtain, applying the Poincaré–inequality, that there exists a constant $C > 0$ such that

$$\begin{aligned}\|\nabla m\|^2_{\mathrm{L}^2(\omega)} &\geq (\|\nabla(m - m^*)\|_{\mathrm{L}^2(\omega)} - \|\nabla m^*\|_{\mathrm{L}^2(\omega)})^2 \\ &\geq C\,\|m - m^*\|^2_{\mathrm{H}^1(\omega)} - 2\,\|m - m^*\|_{\mathrm{H}^1(\omega)}\|\nabla m^*\|_{\mathrm{L}^2(\omega)} + \|\nabla m^*\|^2_{\mathrm{L}^2(\omega)} \\ &\geq C\,\|m - m^*\|^2_{\mathrm{H}^1(\omega)} - \frac{1}{\varepsilon}\,\|m - m^*\|^2_{\mathrm{H}^1(\omega)} - \varepsilon\|\nabla m^*\|^2_{\mathrm{L}^2(\omega)} + \|\nabla m^*\|^2_{\mathrm{L}^2(\omega)} \qquad \forall\,\varepsilon > 0. \end{aligned} \tag{3.29}$$

Therefore, by choosing $\varepsilon > 0$ small enough, (3.28) ensures the existence of constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that

$$I(m, \overline{Q}_{e,s}) \geq C_1\|m - m^*\|^2_{\mathrm{H}^1(\omega)} + C_2 \quad \forall\,(m, \overline{Q}_{e,s}) \in \mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{H}^1(\omega, \mathrm{SO}(3)), \tag{3.30}$$

i.e. the functional $I(m, \overline{Q}_{e,s})$ is bounded from below on $\mathcal{A}$.

Hence, there exists an infimizing sequence $\big\{(m_k, \overline{Q}_k)\big\}_{k=1}^{\infty}$ in $\mathcal{A}$, such that

$$\lim_{k\to\infty} I(m_k, \overline{Q}_k) = \inf\,\big\{I(m, \overline{Q}_{e,s})\,\big|\,(m, \overline{Q}_{e,s}) \in \mathcal{A}\big\}. \tag{3.31}$$

Since we have $I(m^*, \overline{Q}^*_{e,s}) < \infty$, in view of the conditions (3.19), the infimizing sequence $\big\{(m_k, \overline{Q}_k)\big\}_{k=1}^{\infty}$ can be chosen such that

$$I(m_k, \overline{Q}_k) \,\leq\, I(m^*, \overline{Q}^*_{e,s}) \,<\infty\,, \qquad \forall\,k \geq 1. \tag{3.32}$$

Taking into account (3.30) and (3.32) we see that the sequence $\big\{m_k\big\}_{k=1}^{\infty}$ is bounded in $\mathrm{H}^1(\omega, \mathbb{R}^3)$. Then, we can extract a subsequence of $\big\{m_k\big\}_{k=1}^{\infty}$ (not relabeled) which converges weakly in $\mathrm{H}^1(\omega, \mathbb{R}^3)$ and moreover, according to Rellich's selection principle, it converges strongly in $\mathrm{L}^2(\omega, \mathbb{R}^3)$, i.e. there exists an element $\widehat{m} \in \mathrm{H}^1(\omega, \mathbb{R}^3)$ such that

$$m_k \rightharpoonup \widehat{m} \quad \text{in} \quad \mathrm{H}^1(\omega, \mathbb{R}^3), \qquad \text{and} \qquad m_k \to \widehat{m} \quad \text{in} \quad \mathrm{L}^2(\omega, \mathbb{R}^3). \tag{3.33}$$

Corresponding to the fields $(m_k, \overline{Q}_k)$ we consider the strain measures $\mathcal{E}^{(k)}_{m,s}$, $\mathcal{K}^{(k)}_{e,s} \in \mathrm{L}^2(\omega, \mathbb{R}^{3\times3})$. From the coercivity of the internal energy, (3.21) and (3.32) we get

$$C_1\,\|\mathcal{K}^{(k)}_{e,s}\|^2_{\mathrm{L}^2(\omega)} \leq \int_\omega W(\mathcal{E}^{(k)}_{m,s}, \mathcal{K}^{(k)}_{e,s})\,\det[\nabla_x\Theta(0)]\,\mathrm{d}a \leq I(m^*, \overline{Q}^*_{e,s}) + C_2\,\big(\,\|m_k\|_{\mathrm{H}^1(\omega)} + 1\big),$$

where C_1, C_2 are positive constants.

Since $\big\{m_k\big\}_{k=1}^{\infty}$ is bounded in $\mathrm{H}^1(\omega, \mathbb{R}^3)$, it follows from the last inequalities that $\big\{\mathcal{K}^{(k)}_{e,s}\big\}_{k=1}^{\infty}$ is bounded in $\mathrm{L}^2(\omega, \mathbb{R}^{3\times3})$.

For tensor fields P with rows in $\mathrm{H}(\mathrm{curl}\,;\Omega)$, i.e. $P = \big(P^T.e_1\,|\,P^T.e_2\,|\,P^T.e_3\big)^T$ with $(P^T.e_i)^T \in \mathrm{H}(\mathrm{curl}\,;\Omega)$, $i = 1, 2, 3$, we define $\mathrm{Curl}\,P := \big(\mathrm{curl}\,(P^T.e_1)^T\,|\,\mathrm{curl}\,(P^T.e_2)^T\,|\,\mathrm{curl}\,(P^T.e_3)^T\big)^T$. Since $\{\mathcal{K}^{(k)}_{e,s}\}_{k=1}^{\infty}$ is bounded, so is $\big\{\mathrm{axl}(\overline{Q}_k^T\partial_{x_\alpha}\overline{Q}_k)\big\}_{k=1}^{\infty}$, $\alpha = 1, 2$, in $\mathrm{L}^2(\omega, \mathbb{R}^3)$ and it follows that $\overline{Q}_k^T\,\mathrm{Curl}\,\overline{Q}_k$ is bounded. Indeed, using the so-called *wryness tensor* (second order tensor) [38, 18]

$$\Gamma_k := \Big(\mathrm{axl}(\overline{Q}_k^T\,\partial_{x_1}\overline{Q}_k)\,|\,\mathrm{axl}(\overline{Q}_k^T\,\partial_{x_2}\overline{Q}_k)\,|\,0\Big) \in \mathbb{R}^{3\times3}, \tag{3.34}$$

we have (see [38]) the following close relationship (Nye's formula) between the wryness tensor and the dislocation density tensor

$$\alpha_k := \overline{Q}_k^T\,\mathrm{Curl}\,\overline{Q}_k = -\Gamma_k^T + \mathrm{tr}(\Gamma_k)\,\mathbb{1}_3\,, \qquad \text{or equivalently,} \qquad \Gamma_k = -\alpha_k^T + \frac{1}{2}\mathrm{tr}(\alpha_k)\,\mathbb{1}_3\,, \tag{3.35}$$

because $\overline{Q}_k = \overline{Q}_k(x_1, x_2)$. Hence, $\big\{\mathrm{axl}(\overline{Q}_k^T\partial_{x_\alpha}\overline{Q}_k)\big\}_{k=1}^{\infty}$ is bounded if and only if $\overline{Q}_k^T\,\mathrm{Curl}\,\overline{Q}_k$ is bounded. Writing $\mathrm{Curl}\,\overline{Q}_k = \overline{Q}_k\,\overline{Q}_k^T\,\mathrm{Curl}\,\overline{Q}_k$ and using $\|\overline{Q}_k\|^2 = 3$, we deduce that the boundedness of $\overline{Q}_k^T\,\mathrm{Curl}\,\overline{Q}_k$ implies that $\mathrm{Curl}\,\overline{Q}_k$ is bounded. Since the Curl-operator bounds the gradient operator in SO(3), see [38], it follows that $\big\{\partial_{x_\alpha}\overline{Q}_k\big\}_{k=1}^{\infty}$ is bounded in $\mathrm{L}^2(\omega, \mathbb{R}^{3\times3})$, for $\alpha = 1, 2$. Since $\overline{Q}_k \in \mathrm{SO}(3)$ we have $\|\overline{Q}_k\|^2 = 3$ and thus we can infer that the sequence $\{\overline{Q}_k\}_{k=1}^{\infty}$ is bounded in $\mathrm{H}^1(\omega,\mathbb{R}^{3\times 3})$. Hence, there exists a subsequence of $\{\overline{Q}_k\}_{k=1}^{\infty}$ (not relabeled) and an element $\widehat{\overline{Q}}_{e,s}\in \mathrm{H}^1(\omega,\mathbb{R}^{3\times 3})$ with

$$\overline{Q}_k \rightharpoonup \widehat{\overline{Q}}_{e,s} \quad \text{in} \quad \mathrm{H}^1(\omega,\mathbb{R}^{3\times 3}), \qquad \text{and} \qquad \overline{Q}_k \to \widehat{\overline{Q}}_{e,s} \quad \text{in} \quad \mathrm{L}^2(\omega,\mathbb{R}^{3\times 3}). \tag{3.36}$$

Since $\overline{Q}_k\in \mathrm{SO}(3)$ we have

$$\|\overline{Q}_k\widehat{\overline{Q}}_{e,s}^T-\mathbb{1}_3\|_{\mathrm{L}^2(\omega)}=\|\overline{Q}_k(\widehat{\overline{Q}}_{e,s}^T-\overline{Q}_k^T)\|_{\mathrm{L}^2(\omega)}=\|\widehat{\overline{Q}}_{e,s}-\overline{Q}_k\|_{\mathrm{L}^2(\omega)}\to 0,$$

i.e. $\overline{Q}_k\widehat{\overline{Q}}_{e,s}^T\to \mathbb{1}_3$ in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$. On the other hand, we can write

$$\|\overline{Q}_k\widehat{\overline{Q}}_{e,s}^T-\widehat{\overline{Q}}_{e,s}\widehat{\overline{Q}}_{e,s}^T\|_{\mathrm{L}^1(\omega)}=\|(\overline{Q}_k-\widehat{\overline{Q}}_{e,s})\widehat{\overline{Q}}_{e,s}^T\|_{L^1(\omega)}\leq 3\,\|\overline{Q}_k-\widehat{\overline{Q}}_{e,s}\|_{\mathrm{L}^2(\omega)}\,\|\widehat{\overline{Q}}_{e,s}\|_{\mathrm{L}^2(\omega)}\to 0,$$

which means that $\overline{Q}_k\widehat{\overline{Q}}_{e,s}^T\to\widehat{\overline{Q}}_{e,s}\widehat{\overline{Q}}_{e,s}^T$ in $\mathrm{L}^1(\omega,\mathbb{R}^{3\times 3})$. Consequently, we find $\widehat{\overline{Q}}_{e,s}\widehat{\overline{Q}}_{e,s}^T=\mathbb{1}_3$ so that $\widehat{\overline{Q}}_{e,s}$ belongs to $\mathrm{H}^1(\omega,\mathrm{SO}(3))$.

By virtue of the relations $(m_k,\overline{Q}_k)\in\mathcal{A}$ and (3.33), (3.36), we derive that $\widehat{m}=m^*$ on γ_d and $\widehat{\overline{Q}}_{e,s}=\overline{Q}_{e,s}^*$ on γ_d in the sense of traces. Hence, we obtain that the limit pair satisfies $(\widehat{m},\widehat{\overline{Q}}_{e,s})\in\mathcal{A}$.

Let us next construct the limit strain and curvature measures

$$\begin{aligned}
\widehat{\mathcal{E}}_{m,s}:&=\widehat{\overline{Q}}_{e,s}^T(\nabla\widehat{m}|\widehat{\overline{Q}}_{e,s}\nabla_x\Theta(0).e_3)[\nabla_x\Theta(0)]^{-1}-\mathbb{1}_3=Q_0(\widehat{\overline{R}}_s^T\,\nabla\widehat{m}-Q_0^T\nabla y_0|0)[\nabla_x\Theta(0)]^{-1},\\
\widehat{\mathcal{K}}_{e,s}:&=\ (\mathrm{axl}(\widehat{\overline{Q}}_{e,s}^T\,\partial_{x_1}\widehat{\overline{Q}}_{e,s})\,|\,\mathrm{axl}(\widehat{\overline{Q}}_{e,s}^T\,\partial_{x_2}\widehat{\overline{Q}}_{e,s})\,|0)[\nabla_x\Theta(0)]^{-1}\\
&=Q_0\big(\,\mathrm{axl}(\widehat{\overline{R}}_s^T\,\partial_{x_1}\widehat{\overline{R}}_s)-\mathrm{axl}(Q_0^T\partial_{x_1}Q_0)\ \big|\ \mathrm{axl}(\widehat{\overline{R}}_s^T\,\partial_{x_2}\widehat{\overline{R}}_s)-\mathrm{axl}(Q_0^T\partial_{x_2}Q_0)\ \big|\ 0\,\big)[\nabla_x\Theta(0)]^{-1},
\end{aligned}\tag{3.37}$$

where

$$\widehat{\overline{R}}_s(x_1,x_2):=\widehat{\overline{Q}}_{e,s}(x_1,x_2)\,Q_0(x_1,x_2,0)\in\mathrm{SO}(3). \tag{3.38}$$

As shown above, the sequence $\{m_k\}_{k=1}^{\infty}$ is bounded in $\mathrm{H}^1(\omega,\mathbb{R}^3)$. It follows that $\{(\nabla m_k|0)\}_{k=1}^{\infty}$ is bounded in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$. We define

$$\overline{R}_k:=\overline{Q}_k\,Q_0\in\mathrm{SO}(3). \tag{3.39}$$

Then, the sequence $\{\overline{R}_k^T(\nabla m_k|0)\}_{k=1}^{\infty}$ is bounded in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$, since $\overline{R}_k\in\mathrm{SO}(3)$. Consequently, there exists a subsequence (not relabeled) and an element $\xi\in\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$ such that

$$\overline{R}_k^T(\nabla m_k|0)\ \rightharpoonup\ \xi \qquad \text{in} \quad \mathrm{L}^2(\omega,\mathbb{R}^{3\times 3}). \tag{3.40}$$

On the other hand, let $\Phi\in\mathrm{C}_0^\infty(\omega,\mathbb{R}^{3\times 3})$ be an arbitrary test function. Then, using the properties of the scalar product we deduce

$$\begin{aligned}
\int_\omega\big\langle\overline{R}_k^T(\nabla m_k|0)-\widehat{\overline{R}}_s^T(\nabla\widehat{m}|0),\,\Phi\big\rangle\,\mathrm{d}a&=\int_\omega\big\langle\widehat{\overline{R}}_s^T\big((\nabla m_k|0)-(\nabla\widehat{m}|0)\big),\Phi\big\rangle\,\mathrm{d}a+\int_\omega\big\langle\big(\overline{R}_k^T-\widehat{\overline{R}}_s^T\big)(\nabla m_k|0),\Phi\big\rangle\,\mathrm{d}a\\
&=\int_\omega\big\langle(\nabla m_k|0)-(\nabla\widehat{m}|0),\widehat{\overline{R}}_s\Phi\big\rangle\,\mathrm{d}a+\int_\omega\big\langle\overline{R}_k-\widehat{\overline{R}}_s\,,(\nabla m_k|0)\,\Phi^T\big\rangle\mathrm{d}a\\
&\leq\|\overline{R}_k-\widehat{\overline{R}}_s\|_{\mathrm{L}^2(\omega)}\|(\nabla m_k|0)\,\Phi^T\|_{\mathrm{L}^2(\omega)}+\int_\omega\big\langle(\nabla m_k|0)-(\nabla\widehat{m}|0),\widehat{\overline{R}}_s\Phi\big\rangle\,\mathrm{d}a\,.
\end{aligned}\tag{3.41}$$

Since the relations (3.33), (3.36) and $\widehat{\overline{R}}_s\Phi\in\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$ hold, and $\|(\nabla m_k|0)\,\Phi^T\|$ is bounded, we get

$$\int_\omega\big\langle\overline{R}_k^T(\nabla m_k|0),\Phi\big\rangle\,\mathrm{d}a\to\int_\omega\big\langle\widehat{\overline{R}}_s^T(\nabla\widehat{m}|0),\Phi\big\rangle\,\mathrm{d}a,\quad\forall\,\Phi\in\mathrm{C}_0^\infty(\omega,\mathbb{R}^{3\times 3}). \tag{3.42}$$

By comparison of (3.40) and (3.42) we find $\xi = \widehat{\overline{R}}_s^T(\nabla\widehat{m}|0)$, which means that $\overline{R}_k^T(\nabla m_k|0) \rightharpoonup \widehat{\overline{R}}_s^T(\nabla\widehat{m}|0)$ in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times3})$, or equivalently

$$\overline{R}_k^T(\nabla m_k|0) - Q_0^T(\nabla y_0|0) \quad \rightharpoonup \quad \overline{R}_k^T(\nabla\widehat{m}|0) - Q_0^T(\nabla y_0|0) \quad \text{in} \quad \mathrm{L}^2(\omega,\mathbb{R}^{3\times3}). \tag{3.43}$$

Taking into account the hypotheses, we obtain from (3.43) that

$$\mathcal{E}_{m,s}^{(k)} := Q_0(\overline{R}_k^T\,\nabla m_k - Q_0^T\nabla y_0|0)[\nabla_x\Theta(0)]^{-1} \rightharpoonup \widehat{\mathcal{E}}_{m,s} \tag{3.44}$$

in $\mathrm{L}^2(\omega,\mathbb{R}^3)$.

We use now the fact that the sequence $\big\{\mathrm{axl}(\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k)\big\}_{k=1}^\infty$, $\alpha = 1,2$, is bounded in $\mathrm{L}^2(\omega,\mathbb{R}^3)$, since we proved previously that $\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k$ is bounded in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times3})$. Then, there exists a subsequence (not relabeled) and an element $\zeta_\alpha \in \mathrm{L}^2(\omega,\mathbb{R}^3)$, $\alpha = 1,2$, such that

$$\mathrm{axl}(\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k) \rightharpoonup \zeta_\alpha \qquad \text{in} \quad \mathrm{L}^2(\omega,\mathbb{R}^3). \tag{3.45}$$

On the other hand, for any test function $\phi \in \mathrm{C}_0^\infty(\omega,\mathbb{R}^3)$ we can write

$$\begin{aligned}
\int_\omega \big\langle \mathrm{axl}(\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k - \widehat{\overline{R}}_s^T\partial_{x_\alpha}\widehat{\overline{R}}_s)\,,\,\phi\big\rangle_{\mathbb{R}^3}\,\mathrm{d}a &= \frac{1}{2}\int_\omega \big\langle \overline{R}_k^T\partial_{x_\alpha}\overline{R}_k - \widehat{\overline{R}}_s^T\partial_{x_\alpha}\widehat{\overline{R}}_s\,,\,\mathrm{anti}(\phi)\big\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{d}a\\
= \frac{1}{2}\int_\omega \big\langle \widehat{\overline{R}}_s^T\big(\partial_{x_\alpha}\overline{R}_k - \partial_{x_\alpha}\widehat{\overline{R}}_s\big)\,,\,\mathrm{anti}(\phi)\big\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{d}a &+ \frac{1}{2}\int_\omega \big\langle \big(\overline{R}_k^T - \widehat{\overline{R}}_s^T\big)\partial_{x_\alpha}\overline{R}_k\,,\,\mathrm{anti}(\phi)\big\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{d}a\\
\leq \frac{1}{2}\int_\omega \big\langle \partial_{x_\alpha}\overline{R}_k - \partial_{x_\alpha}\widehat{\overline{R}}_s\,,\,\widehat{\overline{R}}_s\,\mathrm{anti}(\phi)\big\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{d}a &+ \frac{1}{2}\|\overline{R}_k - \widehat{\overline{R}}_s\|_{\mathrm{L}^2(\omega)}\,\|\partial_{x_\alpha}\overline{R}_k\,[\mathrm{anti}(\phi)]^T\,\|_{\mathrm{L}^2(\omega)} \to 0,
\end{aligned} \tag{3.46}$$

since $\widehat{\overline{R}}_s\,\mathrm{anti}(\phi) \in \mathrm{L}^2(\omega,\mathbb{R}^{3\times3})$, $\|\partial_{x_\alpha}\overline{R}_k\,[\mathrm{anti}(\phi)]^T\|$ is bounded, and relations (3.36) hold. Consequently, we have

$$\int_\omega \big\langle \mathrm{axl}(\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k)\,,\,\phi\big\rangle_{\mathbb{R}^3}\,\mathrm{d}a \to \int_\omega \big\langle \mathrm{axl}(\widehat{\overline{R}}_s^T\partial_{x_\alpha}\widehat{\overline{R}}_s)\,,\,\phi\big\rangle_{\mathbb{R}^3}\,\mathrm{d}a, \quad \forall\,\phi \in \mathrm{C}_0^\infty(\omega,\mathbb{R}^3), \tag{3.47}$$

and by comparison with (3.45) we deduce that $\zeta_\alpha = \mathrm{axl}(\widehat{\overline{R}}_s{}^T\partial_{x_\alpha}\widehat{\overline{R}}_s)$, i.e.

$$\mathrm{axl}(\overline{R}_k^T\partial_{x_\alpha}\overline{R}_k) - \mathrm{axl}(Q_0\partial_{x_\alpha}Q_0) \quad \rightharpoonup \quad \mathrm{axl}(\widehat{\overline{R}}_s{}^T\partial_{x_\alpha}\widehat{\overline{R}}_s) - \mathrm{axl}(Q_0\partial_{x_\alpha}Q_0) \quad \text{in } \mathrm{L}^2(\omega,\mathbb{R}^3), \tag{3.48}$$

Hence, from (3.20) we derive the convergence

$$\mathcal{K}_{e,s}^{(k)} := Q_0\big(\,\mathrm{axl}(\overline{R}_k^T\partial_{x_1}\overline{R}_k) - \mathrm{axl}(Q_0^T\partial_{x_1}Q_0)\,\big|\,\mathrm{axl}(\overline{R}_k^T\partial_{x_2}\overline{R}_k) - \mathrm{axl}(Q_0^T\partial_{x_2}Q_0)\,\big|\,0\big)[\nabla_x\Theta(0)]^{-1} \rightharpoonup \widehat{\mathcal{K}}_{e,s}. \tag{3.49}$$

In the last step of the proof we use the convexity of the strain energy density W. In view of (3.44) and (3.49), we have

$$\int_\omega W(\widehat{\mathcal{E}}_{m,s},\widehat{\mathcal{K}}_{e,s})\,\det[\nabla_x\Theta(0)]\,\mathrm{d}a \;\leq\; \liminf_{n\to\infty}\int_\omega W(\mathcal{E}_{m,s}^{(k)},\mathcal{K}_{e,s}^{(k)})\,\det[\nabla_x\Theta(0)]\,\mathrm{d}a. \tag{3.50}$$

since W is convex in $(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$. Taking into account the hypotheses (3.18), the continuity of the load potential functions, and the convergence relations $(3.33)_2$ and $(3.36)_2$, we deduce

$$\overline{\Pi}(\widehat{m},\widehat{\overline{Q}}_{e,s}) = \lim_{n\to\infty}\overline{\Pi}(m_k,\overline{Q}_k). \tag{3.51}$$

From (3.50) and (3.51) we get

$$I(\widehat{m},\widehat{\overline{Q}}_{e,s}) \;\leq\; \liminf_{n\to\infty}\, I(m_k,\overline{Q}_k)\,. \tag{3.52}$$

Finally, the relations (3.31) and (3.52) show that

$$I(\widehat{m},\widehat{\overline{Q}}_{e,s}) \;=\; \inf\,\big\{I(m,\overline{Q}_{e,s})\,\big|\,(m,\overline{Q}_{e,s}) \in \mathcal{A}\big\}.$$

Since $(\widehat{m},\widehat{\overline{Q}}_{e,s}) \in \mathcal{A}$, we conclude that $(\widehat{m},\widehat{\overline{Q}}_{e,s})$ is a minimizing solution pair of our minimization problem. ■

The boundary condition on $\overline{Q}_{e,s}$ is not essential in the proof of the above theorem, and that one can prove the existence of minimizers for the minimization problem over a larger admissible set:

Corollary 3.4. [Existence result for the theory including terms up to order $O(h^5)$ without boundary condition on the microrotation field] *Under the hypotheses of Theorem 3.3, the minimization problem* (2.6)–(2.10) *admits at least one minimizing solution pair*

$$(m,\overline{Q}_{e,s}) \in \mathcal{A} = \big\{(m,\overline{Q}_{e,s}) \in \mathrm{H}^1(\omega,\mathbb{R}^3)\times \mathrm{H}^1(\omega,\mathrm{SO}(3)) \;\big|\; m\big|_{\gamma_d} = m^* \big\}\,. \tag{3.53}$$

4 Existence of minimizers for the Cosserat shell model of order $O(h^3)$

In this section we consider only terms up to order $O(h^3)$ in the expression of the energy density. Therefore, we obtain the following two-dimensional minimization problem for the deformation of the midsurface $m:\omega\to\mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s}:\omega\to\mathrm{SO}(3)$ solving on $\omega\subset\mathbb{R}^2$: minimize with respect to $(m,\overline{Q}_{e,s})$ the following functional

$$I(m,\overline{Q}_{e,s})=\int_\omega W^{(h^3)}\big(\mathcal{E}_{m,s},\,\mathcal{K}_{e,s}\big)\,\det(\nabla y_0|n_0)\,\mathrm{d}a-\overline{\Pi}(m,\overline{Q}_{e,s})\,, \tag{4.1}$$

where the shell energy density $W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ is given by

$$\begin{aligned} W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) = &\Big(h+\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big)+\frac{h^3}{12}\,W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\ &-\frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)+\frac{h^3}{6}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big) \\ &+\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\big)+\frac{h^3}{12}W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big), \end{aligned} \tag{4.2}$$

with all the other quantities having the same expressions and interpretations as in the theory up to order $O(h^5)$.

In [21] we have presented a comparison with the the general 6-parameter shell model [19]. While in the previous approaches [19, 10, 11, 7] the dependence of the coefficients upon the curved initial shell configuration is not specified, in our shell model, the constitutive coefficients are deduced from the three-dimensional formulation, while the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem. Another major difference between our model and the previously considered general 6-parameter shell model is that, even in the case of a simplified theory of order $O(h^3)$, additional mixed terms like the membrane–bending part $-\frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)$ and $\frac{h^3}{6}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)$, as well as $W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big)$, are included, which are otherwise difficult to guess. Therefore, an existence proof for the new $O(h^3)$-model is of independent interest. First, we will show

Proposition 4.1. [Coercivity in the theory including terms up to order $O(h^3)$] *Assume that the constitutive coefficients are such that $\mu>0$, $\mu_{\mathrm{c}}>0$, $2\,\lambda+\mu>0$, $b_1>0$, $b_2>0$ and $b_3>0$ and let c_2^+ denotes the smallest eigenvalue of $W_{\mathrm{curv}}(S)$, and c_1^+ and $C_1^+>0$ denote the smallest and the largest eigenvalues of the quadratic form $W_{\mathrm{shell}}(S)$. If the thickness h satisfies one of the following conditions:*

i) $h|\kappa_1|<\frac{1}{2}, \qquad h|\kappa_2|<\frac{1}{2} \quad$ *and* $\quad h^2<\left(\frac{47}{4}\right)^2(5-2\sqrt{6})\frac{c_2^+}{C_1^+}$;

ii) $h|\kappa_1|<\frac{1}{a}\,, \qquad h|\kappa_2|<\frac{1}{a} \quad$ *with* $\quad a>\max\Big\{1+\frac{\sqrt{2}}{2},\frac{1+\sqrt{1+3\frac{C_1^+}{c_1^+}}}{2}\Big\}$,

then $W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ is coercive, in the sense that there exists a constant $a_1^+>0$ such that

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})\;\geq\;a_1^+\,\big(\|\mathcal{E}_{m,s}\|^2+\|\mathcal{K}_{e,s}\|^2\,\big), \tag{4.3}$$

where a_1^+ depends on the constitutive coefficients.

Proof. Using the properties presented in the Appendix, since $\mathrm{B}_{y_0}^2 - 2\,\mathrm{H}\,\mathrm{B}_{y_0} + \mathrm{K}\,\mathrm{A}_{y_0} = 0_3$ and $\mathcal{E}_{m,s}\mathrm{A}_{y_0} = \mathcal{E}_{m,s}$, it follows

$$(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0} = 2\,\mathrm{H}\,\mathcal{E}_{m,s}\mathrm{B}_{y_0} - \mathrm{K}\,\mathcal{E}_{m,s} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}. \tag{4.4}$$

Hence, we have

$$\begin{aligned}
&\Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) + \frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)\\
=&\Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}\mathrm{B}_{y_0}\big) - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) \qquad (4.5)\\
&+ \mathrm{H}\,\frac{h^3}{3}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}\mathrm{B}_{y_0}\big) - \mathrm{K}\,\frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}\big) + \frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)\\
=&\Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) + \frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s}, \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big).
\end{aligned}$$

Using (4.5) and the positive definiteness of the quadratic forms (2.8) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq& \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big) W_{\rm shell}\big(\mathcal{E}_{m,s}\big) + \frac{h^3}{12}\, W_{\rm shell}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&- \frac{h^3}{6}\,|\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\ \mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)| - \frac{h^3}{3}\,|\mathrm{H}|\,|\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\ \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)|\\
&+ \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \frac{h^3}{12}\, W_{\rm curv}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big)\\
\geq& \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{1}{6}\,\big[h\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big)\big]^{\frac{1}{2}}\,\big[h^5\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)\big]^{\frac{1}{2}} \qquad (4.6)\\
&- \frac{1}{3}\,|\mathrm{H}|\,\big[h^2\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big)\big]^{\frac{1}{2}}\,\big[h^4\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)\big]^{\frac{1}{2}} + \frac{h^3}{12}\, W_{\rm shell}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&+ \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \frac{h^3}{12}\, W_{\rm curv}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big).
\end{aligned}$$

In view of the arithmetic-geometric mean inequality it follows

$$\begin{aligned}
W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq& \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{1}{12}\,\varepsilon\, h\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{1}{12\,\varepsilon}\, h^5\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)\\
&- \frac{1}{6}\,\delta\,|\mathrm{H}|\, h^2\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) - \frac{1}{6\,\delta}\,|\mathrm{H}|\, h^4\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) + \frac{h^3}{12}\, W_{\rm shell}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&+ \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \frac{h^3}{12}\, W_{\rm curv}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big) \quad \forall\,\varepsilon > 0 \text{ and } \delta > 0. \qquad (4.7)
\end{aligned}$$

Using the inequalities $-h^2|K| > -\frac{1}{4}$ and $-h\,|H| > -\frac{1}{2}$, we obtain

$$\begin{aligned}
W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq& \frac{h}{12}\Big(\frac{47}{4} - \delta - \varepsilon\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big) + \frac{h^3}{12}\, W_{\rm shell}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)\\
&- \frac{1}{12\,\delta}\, h^3\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) - \frac{1}{12\,\varepsilon}\, h^5\, W_{\rm shell}\big(\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big) \qquad (4.8)\\
&+ \frac{47\,h}{48}\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \frac{h^3}{12}\, W_{\rm curv}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\,\big) \quad \forall\,\varepsilon > 0 \text{ and } \delta > 0.
\end{aligned}$$

In view of (3.3) and (2.12) and since the Frobenius norm is sub-multiplicative, we deduce

$$\begin{aligned}
W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq& \frac{h}{12}\Big(\frac{47}{4} - \delta - \varepsilon\Big)\, c_1^+\,\|\mathcal{E}_{m,s}\|^2 + \frac{h^3}{12}\, c_1^+\,\|\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2\\
&- \frac{1}{12\,\delta}\, h^3\, C_1^+\|\mathrm{C}_{y_0}\|^2\,\|\mathcal{K}_{e,s}\|^2 - \frac{1}{12\,\varepsilon}\, h^5\, C_1^+\|\mathrm{C}_{y_0}\|^2\,\|\mathcal{K}_{e,s}\,\mathrm{B}_{y_0}\|^2 \qquad (4.9)\\
&+ \frac{47\,h}{48}\, c_2^+\,\|\mathcal{K}_{e,s}\|^2 + \frac{h^3}{12}\, c_2^+\,\|\mathcal{K}_{e,s}\mathrm{B}_{y_0}\|^2 \quad \forall\,\varepsilon > 0 \text{ and } \delta > 0 \text{ such that } \frac{47}{4} > \delta + \varepsilon.
\end{aligned}$$

Since $\|C_{y_0}\|^2 = 2$, the estimate (4.9) becomes

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq \frac{h}{12}\Big(\frac{47}{4}-\delta-\varepsilon\Big)\; c_1^+\|\mathcal{E}_{m,s}\|^2+\frac{h^3}{12}\,c_1^+\|\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2$$
$$+\,C_1^+\frac{h^3}{6}\Big(\frac{47}{8}\,\frac{1}{h^2}\,\frac{c_2^+}{C_1^+}-\frac{1}{\delta}\Big)\|\mathcal{K}_{e,s}\|^2+C_1^+\frac{h^5}{6}\,\Big(\frac{1}{2}\,\frac{1}{h^2}\,\frac{c_2^+}{C_1^+}-\frac{1}{\varepsilon}\Big)\|\mathcal{K}_{e,s}\mathrm{B}_{y_0}\|^2, \tag{4.10}$$

for all $\varepsilon > 0$ and $\delta > 0$ such that $\dfrac{47}{4} > \delta+\varepsilon$. According to the properties presented in the Appendix, we have

$$\|B_{y_0}\|^2 = \big\langle B_{y_0}^2,\mathbb{1}\big\rangle = 2\,\mathrm{H}\,\big\langle B_{y_0},\mathbb{1}\big\rangle - \mathrm{K}\,\big\langle A_{y_0},\mathbb{1}\big\rangle = 4\,\mathrm{H}^2-2\,\mathrm{K}. \tag{4.11}$$

Therefore, from (4.10) it follows

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq \frac{h}{12}\Big(\frac{47}{4}-\delta-\varepsilon\Big)\,c_1^+\|\mathcal{E}_{m,s}\|^2+\frac{h^3}{12}\,c_1^+\|\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2 \tag{4.12}$$
$$+\,h\,\Big[\frac{47}{48}\,c_2^+-\frac{1}{6\,\delta}\,h^2\,C_1^+\;-\frac{1}{6\,\varepsilon}\,h^4\,C_1^+\,(4\,\mathrm{H}^2-2\,\mathrm{K})\Big]\|\mathcal{K}_{e,s}\|^2+h^3\,\frac{1}{12}\,c_2^+\|\mathcal{K}_{e,s}\mathrm{B}_{y_0}\|^2.$$

Using again that h is small, we obtain $-h^2\,(4\,\mathrm{H}^2-2\,\mathrm{K}) > -\frac{3}{2}$ and

$$\begin{aligned}W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) &\geq \frac{h}{12}\Big(\frac{47}{4}-\delta-\varepsilon\Big)\,c_1^+\|\mathcal{E}_{m,s}\|^2+\frac{h^3}{12}\,c_1^+\|\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2 \\ &\quad+\frac{h^3}{12}\,C_1^+\,\Big[\frac{47}{4}\,\frac{c_2^+}{h^2\,C_1^+}-\frac{2}{\delta}\;-\frac{3}{\varepsilon}\Big]\|\mathcal{K}_{e,s}\|^2+h^3\,\frac{1}{12}\,c_2^+\|\mathcal{K}_{e,s}\mathrm{B}_{y_0}\|^2 \\ &\geq \frac{h}{12}\Big(\frac{47}{4}-\delta-\varepsilon\Big)\,c_1^+\|\mathcal{E}_{m,s}\|^2+\frac{h^3}{12}\,C_1^+\,\Big[\frac{47}{4}\,\frac{c_2^+}{h^2\,C_1^+}-\frac{2}{\delta}\;-\frac{3}{\varepsilon}\Big]\|\mathcal{K}_{e,s}\|^2.\end{aligned} \tag{4.13}$$

We consider $\delta = \gamma\,\varepsilon$ and we choose $\delta > 0$ and $\gamma > 0$ such that $\frac{47}{4\,(1+\gamma)} > \varepsilon > \frac{2+3\,\gamma}{\gamma}\,\frac{4}{47}\,\frac{h^2\,C_1^+}{c_2^+}$. This choice of the variable ε is possible if and only if $\left(\frac{47}{4}\right)^2\,\frac{\gamma}{(2+3\,\gamma)\,(1+\gamma)} > \frac{h^2\,C_1^+}{c_2^+}$. At this point we use that $\max_{\gamma>0}\frac{\gamma}{(2+3\,\gamma)\,(1+\gamma)} = 5-2\sqrt{6}$, and we take $\gamma = \sqrt{\frac{2}{3}}$. Note that the considered values for γ and ε assure that the condition $\dfrac{47}{4} > \delta+\varepsilon$ is automatically satisfied. Hence, we arrive at the following condition on the thickness h:

$$h^2 < \left(\frac{47}{4}\right)^2(5-2\sqrt{6})\frac{c_2^+}{C_1^+} \approx 13.94\frac{c_2^+}{C_1^+}, \tag{4.14}$$

which proves the coercivity if the condition i) is satisfied.

Next, we consider coercivity for condition ii). Under the hypotheses of the theorem, using also the positive definiteness of the quadratic forms (2.8) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) &\geq \Big(h+\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big)-\frac{1}{3}\,|\mathrm{H}|\,\,\big[h^2\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big)\big]^{\frac{1}{2}}\,\big[h^4\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)\big]^{\frac{1}{2}} \\ &\quad+\frac{h^3}{12}\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)-\frac{1}{6}\,\Big[hW_{\text{shell}}\big(\mathcal{E}_{m,s}\big)\Big]^{\frac{1}{2}}\,\Big[h^5W_{\text{shell}}\big((\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)\Big]^{\frac{1}{2}} \\ &\quad+\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\text{curv}}\big(\mathcal{K}_{e,s}\big).\end{aligned} \tag{4.15}$$

Using the arithmetic-geometric mean inequality in the previous estimate, it follows

$$\begin{aligned}W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) &\geq \Big(h+\mathrm{K}\,\frac{h^3}{12}-\frac{h^2}{6}\varepsilon\,|\mathrm{H}|\Big)\;W_{\text{shell}}\big(\mathcal{E}_{m,s}\big)+\Big(\frac{h^3}{12}\;-\frac{h^4}{6\,\varepsilon}\;|\mathrm{H}|\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big) \\ &\quad-\frac{h}{12}\delta\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big)-\frac{h^5}{12\,\delta}W_{\text{shell}}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big) \\ &\quad+\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\text{curv}}\big(\mathcal{K}_{e,s}\big).\end{aligned} \tag{4.16}$$

We choose $\delta = 8$ and $\varepsilon = 2$ to obtain that

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq h\left[\frac{1}{3} - \mathrm{K}\,\frac{h^2}{12} - \frac{h}{3}\,|\mathrm{H}|\right] W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big) + \frac{h^3}{12}\Big(1 - h\,|\mathrm{H}|\Big)\, W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)$$
$$- \frac{h^5}{96}\, W_{\mathrm{shell}}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\,\big) + \Big(h - |\mathrm{K}|\,\frac{h^3}{12}\Big)\, W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\big). \tag{4.17}$$

Let us consider $a > 0$ and impose $h\,|H| < \frac{1}{a}$, $h^2\,|K| < \frac{1}{a^2}$. Therefore, using (2.12) and since the Frobenius norm is sub-multiplicative we deduce

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq h\,\frac{4\,a^2 - 4\,a - 1}{12\,a^2}\; c_1^+\,\|\mathcal{E}_{m,s}\|^2 + h\,\frac{12\,a^2 - 1}{12\,a^2}\; c_2^+\,\|\mathcal{K}_{e,s}\|^2 \tag{4.18}$$
$$+ \frac{h^3}{12}\,\frac{a-1}{a}\; c_1^+\,\|\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\|^2 - \frac{h^5}{96}\; C_1^+\,\|\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\|^2\,\|\mathrm{B}_{y_0}\|^2.$$

Moreover, using (4.11) we deduce $-h^2\|B_{y_0}\|^2 > -\frac{6}{a^2}$ and the inequality

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq h\,\frac{4\,a^2 - 4\,a - 1}{12\,a^2}\; c_1^+\,\|\mathcal{E}_{m,s}\|^2 + h\,\frac{12\,a^2 - 1}{12\,a^2}\; c_2^+\,\|\mathcal{K}_{e,s}\|^2 \tag{4.19}$$
$$+ \frac{h^3}{12} C_1^+\,\frac{a-1}{a}\,\left[\frac{c_1^+}{C_1^+} - \frac{3}{4\,a(a-1)}\right] \|\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\|^2\,.$$

Hence, choosing $a > 1 + \frac{\sqrt{2}}{2}$ we assure that $\frac{4\,a^2 - 4\,a - 1}{12\,a^2} > 0$, $\frac{a-1}{a} > 0$ and $\frac{12\,a^2-1}{12\,a^2} > 0$. A suitable $a > 1 + \frac{\sqrt{2}}{2}$ should satisfy $\frac{c_1^+}{C_1^+} - \frac{3}{4\,a(a-1)} > 0$, which is true if a is such that $a > \frac{1+\sqrt{1+3\frac{C_1^+}{c_1^+}}}{2} > 1$. Therefore, if $a > \max\left\{1 + \frac{\sqrt{2}}{2}, \frac{1+\sqrt{1+3\frac{C_1^+}{c_1^+}}}{2}\right\}$, then inequality (4.19) yields

$$W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}) \geq h\,\frac{4\,a^2 - 4\,a - 1}{12\,a^2}\; c_1^+\,\|\mathcal{E}_{m,s}\|^2 + h\,\frac{12\,a^2 - 1}{12\,a^2}\; c_2^+\,\|\mathcal{K}_{e,s}\|^2. \qquad \blacksquare$$

Corollary 4.2. [Uniform convexity for the theory including terms up to order $O(h^3)$] *Under the hypotheses of Proposition 4.1, the energy density $W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$ is uniformly convex in $(\mathcal{E}_{m,s},\mathcal{K}_{e,s})$, i.e. there exists a constant $a_1^+ > 0$ such that*

$$D^2\, W^{(h^3)}(\mathcal{E}_{m,s},\mathcal{K}_{e,s}).\,[(H_1,H_2),(H_1,H_2)] \geq a_1^+(\|H_1\|^2 + \|H_2\|^2) \qquad \forall\, H_1, H_2 \in \mathbb{R}^{3\times 3}. \tag{4.20}$$

Proof. See Corollary 3.2. ■

Therefore, an existence result similar to Theorem 3.3 holds true for the theory including terms up to order $O(h^3)$:

Theorem 4.3. [Existence result for the theory including terms up to order $O(h^3)$] *Assume that the external loads satisfy the conditions*

$$f \in \mathrm{L}^2(\omega,\mathbb{R}^3), \qquad t \in \mathrm{L}^2(\gamma_t,\mathbb{R}^3), \tag{4.21}$$

the boundary data satisfy the conditions

$$m^* \in \mathrm{H}^1(\omega,\mathbb{R}^3), \qquad \overline{Q}_{e,s}^{*} \in \mathrm{H}^1(\omega,\mathrm{SO}(3)), \tag{4.22}$$

and that the following conditions concerning the initial configuration are fulfilled: $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a continuous injective mapping and

$$y_0 \in \mathrm{H}^1(\omega,\mathbb{R}^3), \qquad Q_0(0) \in \mathrm{H}^1(\omega,\mathrm{SO}(3)), \qquad \nabla_x\Theta(0) \in \mathrm{L}^\infty(\omega,\mathbb{R}^{3\times 3}), \qquad \det[\nabla_x\Theta(0)] \geq\, a_0 > 0\,, \tag{4.23}$$

where a_0 is a constant. Assume that the constitutive coefficients are such that $\mu > 0$, $\mu_c > 0$, $2\,\lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$. Then, if the thickness h satisfies at least one of the following conditions:

i) $h|\kappa_1| < \frac{1}{2}, \qquad h|\kappa_2| < \frac{1}{2} \quad$ *and* $\quad h^2 < \left(\frac{47}{4}\right)^2 (5-2\sqrt{6})\frac{c_2^+}{C_1^+}$;

ii) $h|\kappa_1| < \frac{1}{a}\,, \qquad h|\kappa_2| < \frac{1}{a} \quad$ *with* $\quad a > \max\Big\{1+\frac{\sqrt{2}}{2}, \frac{1+\sqrt{1+3\frac{C_1^+}{c_1^+}}}{2}\Big\}$,

where c_2^+ *denotes the smallest eigenvalue of* $W_{\rm curv}(S)$, *and* c_1^+ *and* $C_1^+>0$ *denote the smallest and the biggest eigenvalues of the quadratic form* $W_{\rm shell}(S)$, *the minimization problem corresponding to the energy density defined by* (4.1) *and* (4.2) *admits at least one minimizing solution pair* $(m,\overline{Q}_{e,s})\in\mathcal{A}$.

Corollary 4.4. [Existence result for the theory including terms up to order $O(h^3)$ without boundary condition on the microrotation field] *Under the hypotheses of Theorem 4.3, the minimization problem corresponding to the energy density defined by* (4.1) *and* (4.2) *admits at least one minimizing solution pair*

$$(m,\overline{Q}_{e,s})\in\mathcal{A}=\big\{(m,\overline{Q}_{e,s})\in{\rm H}^1(\omega,\mathbb{R}^3)\times{\rm H}^1(\omega,{\rm SO}(3))\ \big|\ m\big|_{\gamma_d}=m^*\big\}\,. \tag{4.24}$$

5 Final comments

Having the deformation of the midsurface $m:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s}:\omega\subset\mathbb{R}^2\to{\rm SO}(3)$ solving on ω the minimization (two-dimensional) problem, we get the approximation of the deformation of the initial three-dimensional body using the following *6-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s:\Omega_h\subset\mathbb{R}^3\to\mathbb{R}^3$ of the shell-like structure [21]

$$\varphi_s(x_1,x_2,x_3)\,=m(x_1,x_2)+\left(x_3\varrho_m(x_1,x_2)+\frac{x_3^2}{2}\varrho_b(x_1,x_2)\right)\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0).e_3\,, \tag{5.1}$$

where $\varrho_m,\ \varrho_b:\omega\subset\mathbb{R}^2\to\mathbb{R}$ allow in principal for symmetric thickness stretch ($\varrho_m\neq 1$) and asymmetric thickness stretch ($\varrho_b\neq 0$) about the midsurface and they are given by

$$\begin{aligned}
\varrho_m\,&=1-\frac{\lambda}{\lambda+2\,\mu}[\langle\overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2]\,=\,1-\frac{\lambda}{\lambda+2\mu}{\rm tr}(\mathcal{E}_{m,s})\,,\\
\varrho_b\,&=-\frac{\lambda}{\lambda+2\,\mu}\langle\overline{Q}_{e,s}^T(\nabla(\,\overline{Q}_{e,s}\nabla_x\Theta(0).e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\\
&\quad+\frac{\lambda}{\lambda+2\,\mu}\langle\overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\,=\,-\frac{\lambda}{\lambda+2\mu}{\rm tr}[{\rm C}_{y_0}\mathcal{K}_{e,s}+\mathcal{E}_{m,s}{\rm B}_{y_0}].
\end{aligned} \tag{5.2}$$

Obviously, if we know the total microrotation $\overline{R}_s(x_1,x_2)=\overline{Q}_{e,s}(x_1,x_2)\,Q_0(x_1,x_2,0)\in{\rm SO}(3)$, then we know the microrotation $\overline{R}_\xi$ of the parental three-dimensional problem, since we assume it is independent of x_3.

It is noteworthy that the existence result in the $O(h^3)$-model is not simply the truncated version of the existence result for the $O(h^5)$-model. Both existence results require uniformly positive constitutive parameters, in particular we need to assume that the Cosserat couple modulus $\mu_{\rm c}>0$. In the interesting no-drill limit case $\mu_{\rm c}\equiv 0$, we would need new generalized Korn's inequalities [27, 42, 43, 39], which couple the smoothness of the rotation field $\overline{R}_s$ with the coercivity with respect to the deformation m, in the sense that

$$\|\overline{R}_s^T\,(\nabla m|0)+(\nabla m|0)^T\overline{R}_s\|^2_{{\rm L}^2(\omega)}\geq c^+\|m\|^2_{{\rm H}^1(\omega)}\,. \tag{5.3}$$

However, such an estimate is currently only known for $\overline{R}_s\in C(\overline{\omega},{\rm SO}(3))$, but the elastic shell energy only assures $\overline{R}_s\in{\rm H}^1(\omega,{\rm SO}(3))$. More research is needed in this direction.

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Appendix. Properties of the considered tensors

In this paper we use some properties of the tensors involved in the variational formulation of the shell model [21].

Proposition A.1. *The following identities are satisfied :*

i) $\mathrm{tr}[\mathrm{A}_{y_0}] = 2, \quad \det[\mathrm{A}_{y_0}] = 0; \quad \mathrm{tr}[\mathrm{B}_{y_0}] = 2\,\mathrm{H}\,, \quad \det[\mathrm{B}_{y_0}] = 0,$

ii) B_{y_0} *satisfies the equation of Cayley-Hamilton type* $\mathrm{B}_{y_0}^2 - 2\,\mathrm{H}\,\mathrm{B}_{y_0} + \mathrm{K}\,\mathrm{A}_{y_0} = 0_3;$

iii) $\mathrm{A}_{y_0}\mathrm{B}_{y_0} = \mathrm{B}_{y_0}\mathrm{A}_{y_0} = \mathrm{B}_{y_0}\,, \quad \mathrm{A}_{y_0}^2 = \mathrm{A}_{y_0};$

iv) $\mathrm{C}_{y_0} \in \mathfrak{so}(3), \quad \mathrm{C}_{y_0}^2 = -\mathrm{A}_{y_0}$ *and it has the simplified form* $\mathrm{C}_{y_0} := Q_0(0)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_0^T(0) \in \mathfrak{so}(3).$

v) $\overline{Q}_{e,s}^T\,(\nabla[\overline{Q}_{e,s}\nabla_x\Theta(0).e_3]\,|\,0)\,[\nabla_x\Theta(0)]^{-1} = \mathrm{C}_{y_0}\mathcal{K}_{e,s} - \mathrm{B}_{y_0};$

vi) $\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{A}_{y_0} \;=\; \mathrm{C}_{y_0}\mathcal{K}_{e,s}$;

vii) $\mathcal{E}_{m,s}\mathrm{A}_{y_0} \;=\; \mathcal{E}_{m,s}$.

Proof. For the proof of this proposition we refer to [21]. Here, we prove only the third identity of iv).

We have $[\nabla_x\Theta(x_3)].e_3 \;=\; n_0$. Let us recall that $X \in \mathrm{GL}^+(3)$ satisfies the ***G**eneralized **K**irchhoff **C**onstraint* (GKC) [29] if $X \in \mathrm{GKC} := \{X \in \mathrm{GL}^+(3) \mid X^T X.e_3 \;=\; \varrho^2 e_3,\ \varrho \in \mathbb{R}^+\}$. For all $X \in \mathrm{GKC}$ with the polar decomposition $X = R\,U_0$, if follows that $U_0 \in \mathrm{GKC}$. In view of this property and $\nabla\Theta(x_3) = Q_0(x_3)U_0(x_3)$, it follows[5] $U_0(x_3) \;=\; \begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{pmatrix} \in \mathrm{Sym}^+(3)$. Since $\det Q_0 \;=\; 1$, we deduce

$$\mathrm{C}_{y_0} \;=\; \mathrm{Cof}(\nabla_x\Theta(0)) \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} [\nabla_x\Theta(0)]^{-1} \;=\; Q_0(0)\,(\det U_0(0))\,U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} U_0^{-1}(0)\,Q_0^T(0). \tag{A.1}$$

Direct computations give us $\begin{pmatrix} a & x & 0\\ x & b & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \begin{pmatrix} a & x & 0\\ x & b & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} 0 & ab-x^2 & 0\\ x^2-ab & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ and $\det \begin{pmatrix} a & x & 0\\ x & b & 0\\ 0 & 0 & 1\end{pmatrix}^{-1} = \dfrac{1}{ab-x^2}$. Using these calculation in (A.1), we obtain

$$(\det U_0(0))\,U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} U_0^{-1}(0) \;=\; \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{A.2}$$

Hence, the alternator tensor has the representation given in iv). ■

[5]Here, $*$ denotes quantities having expressions which are not relevant for our calculations.