



OCTAV MAYER INSTITUTE OF MATHEMATICS
ROMANIAN ACADEMY, IAȘI BRANCH
Bd. Carol I nr. 10, 700506, Iași, Romania

PREPRINT SERIES OF THE
OCTAV MAYER INSTITUTE OF MATHEMATICS

ON A METRIC CONDITION ON SETS AND ITS
CONSEQUENCES IN OPTIMIZATION

Marius Durea

Faculty of Mathematics, “Alexandru Ioan Cuza” University, Iași, Romania
Octav Mayer Institute of Mathematics of the Romanian Academy, Iași,
Romania

Diana Plop

Faculty of Mathematics, “Alexandru Ioan Cuza” University, Iași, Romania

Radu Strugariu

Department of Mathematics, “Gheorghe Asachi” Technical University, Iași,
Romania

Nr. 13-2020

ON A METRIC CONDITION ON SETS AND ITS CONSEQUENCES IN OPTIMIZATION

by

MARIUS DUREA¹, DIANA MAXIM², and RADU STRUGARIU³

Abstract: We study a metric inequality on sets that ensures the applicability of standard necessary optimality conditions for constrained optimization problems when a new constraint is added. We compare this condition with other constraint qualification conditions in literature and, due to its metric nature, we apply it to nonsmooth optimization problems in order to perform first a penalization and then to give optimality conditions in terms of generalized differentiability.

Keywords: metric conditions · constraint qualification conditions · penalization · optimality conditions

Mathematics Subject Classification (2010): 54C60 · 46G05 · 90C46

1 Introduction and preliminaries

The starting point of the investigation we propose in this work is the question we briefly present below. Generally, if one considers a constrained optimization problem, in order to write necessary optimality conditions, some constraint qualification conditions are necessary. Suppose that we add to the current system of constraints a new constraint. Of course, the problem can dramatically change, and even if the initial system satisfies a constraint qualification condition, the new system can fail to do so. We asked ourselves if one can give a condition that links the old system of constraints and the new constraint in such a way that the optimality conditions apply for the new problem, without checking a constraint qualification condition for the whole new system of constraints. Actually, we started by asking this question in the case of a smooth optimization problem with inequalities constraints, and then we observed that in order to keep the requirements as minimal as possible, we arrive at a metric inequality that naturally comes into play for other types of optimization problems, including nonsmooth ones, and for some penalization results of Clarke's type, as well.

The description we give next of how the paper is organized allows us to underline more details about the ideas on which it is based.

¹Faculty of Mathematics, "Alexandru Ioan Cuza" University, Bd. Carol I, nr. 11, 700506 – Iași, Romania, e-mail: durea@uaic.ro, and "Octav Mayer" Institute of Mathematics of the Romanian Academy, Iași, Romania.

²Faculty of Mathematics, "Alexandru Ioan Cuza" University, Bd. Carol I, nr. 11, 700506 – Iași, Romania, e-mail: plop.diana.elena@gmail.com

³Department of Mathematics, "Gheorghe Asachi" Technical University, Bd. Carol I, nr. 11, 700506 – Iași, Romania, e-mail: rstrugariu@tuiasi.ro

In the second section, we present some basic facts about the cost in terms of assumptions needed for the addition of a new constraint in a smooth scalar optimization problem with inequalities. The third section deals with a metric condition (in fact, a metric inequality) designed to fill the gap between the assumptions used before the addition of a new constraint and the ones needed for successful implementation of the optimality conditions into the new problem. We perform, by several examples, a comparison of this metric condition and the usual Mangasarian-Fromowitz condition. Moreover, we see that this inequality is equivalent, but simpler to check, in comparison to other similar conditions used in literature. Another important feature of our condition is the fact that it can be employed in nonsmooth settings, for problems much more general than those we started with. Section four is twofold. On one hand, we apply some facts collected in the previous sections to directional Pareto minima in order to get necessary optimality conditions, and, on the other hand, we employ the general pattern of the metric inequality under study to penalize scalar nonsmooth optimization problems with multiple constraints. The latter approach allows us to derive necessary optimality conditions in terms of limiting (Mordukhovich) generalized differentiation techniques for the problems under consideration. The paper ends with some concluding comments where, in particular, we briefly describe some possible continuations of this work.

The notation is fairly standard. If X is a normed vector space, then we denote by $B(x, r)$, $D(x, r)$ and $S(x, r)$ the open ball, the closed ball and the sphere of center $x \in X$ and radius $r > 0$, respectively. For a set $A \subset X$, we denote by $\text{int } A$, $\text{cl } A$, $\text{bd } A$ its topological interior, closure and boundary, respectively. The cone generated by A is designated by $\text{cone } A$, and the convex hull of A is $\text{conv } A$. The distance from a point $x \in X$ to a nonempty set $A \subset X$ is $d(x, A) := \inf \{\|x - a\| \mid a \in A\}$ and the distance function to A is $d_A : X \rightarrow \mathbb{R}$ given by $d_A(x) := d(x, A)$. The topological dual of X is X^* , and the negative polar of A is

$$A^- = \{x^* \in X^* \mid x^*(a) \leq 0, \forall a \in A\}.$$

The positive polar of A is $A^+ := -A^-$. Of course, $A^- = (\text{cone } A)^-$.

Let D be a nonempty subset of X and $\bar{x} \in X$. The first order Bouligand tangent cone to D at \bar{x} is the set

$$T_B(D, \bar{x}) = \{u \in X \mid \exists(t_n) \downarrow 0, \exists(u_n) \rightarrow u, \forall n \in \mathbb{N}, \bar{x} + t_n u_n \in D\}$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0, \infty)$ and $t_n \rightarrow 0$. The first order Ursescu tangent cone to D at \bar{x} is the set

$$T_U(D, \bar{x}) = \{u \in X \mid \forall(t_n) \downarrow 0, \exists(u_n) \rightarrow u, \forall n \in \mathbb{N}, \bar{x} + t_n u_n \in D\}.$$

The first order Dubovitskij-Miljutin tangent set to D at \bar{x} is the set

$$T_{DM}(D, \bar{x}) = \{u \in X \mid \forall(t_n) \downarrow 0, \forall(u_n) \rightarrow u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \bar{x} + t_n u_n \in D\}.$$

The Bouligand and Ursescu tangent cones are closed sets, and $T_U(D, \bar{x}) \subset T_B(D, \bar{x})$. The fact that $T_B(D, \bar{x}) = X \setminus T_{DM}(X \setminus D, \bar{x})$ shows that the Dubovitskij-Miljutin tangent set to D at \bar{x} is open. Moreover, for $*$ $\in \{B, U, DM\}$ we have $T_*(D, \bar{x}) = T_*(\text{cl } D, \bar{x})$. If $\bar{x} \in \text{int } A$, then $T_*(D, \bar{x}) = T_*(D \cap A, \bar{x})$. It is well known that $\bar{x} \in \text{int } D$ if and only if $T_{DM}(D, \bar{x}) = X$ and $\bar{x} \in \text{cl } D$ if and only if $T_B(D, \bar{x}) \neq \emptyset$.

Now, we briefly collect some basic facts concerning the limiting generalized calculus (see [12]). The effectiveness of this calculus relies on the concept of normal cone and its main features hold

on Asplund spaces, which represent a special class of Banach spaces: X is Asplund if, and only if, every continuous convex function on any open convex set $U \subset X$ is Fréchet differentiable at the points of a dense G_δ -subset of U . A very important property of Asplund spaces is that every bounded sequence of the topological dual admits a w^* -convergent subsequence.

Take a nonempty subset S of the Asplund space X and pick $x \in S$. Then the Fréchet normal cone to S at x is

$$\widehat{N}(S, x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u - x)}{\|u - x\|} \leq 0 \right\},$$

where $u \xrightarrow{S} x$ means that $u \rightarrow x$ and $u \in S$.

Let $\bar{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at \bar{x} is

$$N(S, \bar{x}) := \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\}.$$

If $S \subset X$ is a convex set, then

$$N(S, \bar{x}) = \{x^* \in X^* \mid x^*(x - \bar{x}) \leq 0, \forall x \in S\}$$

and coincides with the negative polar of $T_B(S, \bar{x})$.

Let $F : X \rightrightarrows Y$ be a set-valued map between the Asplund spaces X and Y , and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Then the normal coderivative of F at (\bar{x}, \bar{y}) is the set-valued map $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\text{Gr } F, (\bar{x}, \bar{y}))\}.$$

As usual, when $F := f$ is a function, since $\bar{y} \in F(\bar{x})$ means $\bar{y} = f(\bar{x})$, we write $D^*f(\bar{x})$ for $D^*f(\bar{x}, \bar{y})$.

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be finite at $\bar{x} \in X$ and lower semicontinuous around \bar{x} ; the Fréchet subdifferential of f at \bar{x} is given by

$$\widehat{\partial}f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\text{epi } f, (\bar{x}, f(\bar{x})))\},$$

where $\text{epi } f$ denotes the epigraph of f ; similarly, the basic (or limiting, or Mordukhovich) subdifferential of f at \bar{x} is given by

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\text{epi } f, (\bar{x}, f(\bar{x})))\}.$$

One always has $\widehat{\partial}f(\bar{x}) \subset \partial f(\bar{x})$. Note that a generalized Fermat rule holds: if \bar{x} is a local minimum point for f then $0 \in \widehat{\partial}f(\bar{x})$.

It is well-known that if A is a closed set and $\bar{x} \in A$, then

$$\begin{aligned} \widehat{\partial}d_A(\bar{x}) &= \widehat{N}(A, \bar{x}) \cap D(0, 1), \\ \widehat{N}(A, \bar{x}) &= \bigcup_{\lambda > 0} \lambda \widehat{\partial}d_A(\bar{x}), \\ \partial d_A(\bar{x}) &\subset N(A, \bar{x}) \cap D(0, 1). \end{aligned}$$

If f is a convex function, then both $\widehat{\partial}f(\bar{x})$ and $\partial f(\bar{x})$ coincide with the Fenchel subdifferential.

Moreover, the next calculus rule holds for the Fréchet subdifferential of the difference of mappings (see [13, Theorem 3.1]): if $f_1, f_2 : X \rightarrow \overline{\mathbb{R}}$ are finite at \bar{x} and $\widehat{\partial}f_2(\bar{x}) \neq \emptyset$, then

$$\widehat{\partial}(f_1 - f_2)(\bar{x}) \subset \bigcap_{x^* \in \widehat{\partial}f_2(\bar{x})} \left[\widehat{\partial}f_1(\bar{x}) - x^* \right] \subset \widehat{\partial}f_1(\bar{x}) - \widehat{\partial}f_2(\bar{x}). \quad (1.1)$$

2 Adding a new constraint

We start by illustrating on a smooth optimization problem the main question we deal with in this paper. Let $f, g : X \rightarrow \mathbb{R}$ be continuously differentiable functions. Consider the basic optimization problem

$$\min f(x), \text{ subject to } g(x) \leq 0,$$

and let $\bar{x} \in X$ be an optimal solution of this problem. The first-order necessary optimality condition is

$$\nabla f(\bar{x})(u) \geq 0, \quad \forall u \in T_B(M_g, \bar{x}), \quad (2.1)$$

where

$$M_g := \{x \in X \mid g(x) \leq 0\}$$

is the set of feasible points. We see that one important issue is to describe the cone $T_B(M_g, \bar{x})$. Clearly, if the constraint is not active at the feasible point \bar{x} (that is, $g(\bar{x}) < 0$), then $T_B(M_g, \bar{x}) = X$ and (2.1) becomes $\nabla f(\bar{x}) = 0$ (Fermat's Theorem).

Otherwise, if the constraint is active at \bar{x} , i.e., $g(\bar{x}) = 0$, we have to suppose that $\nabla g(\bar{x}) \neq 0$ in order to obtain that

$$T_B(M_g, \bar{x}) = T_U(M_g, \bar{x}) = \text{cl } T_{DM}(M_g, \bar{x}) = \{u \in X \mid \nabla g(\bar{x})(u) \leq 0\}. \quad (2.2)$$

In order to show this, observe first that

$$\text{cl } T_{DM}(M_g, \bar{x}) \subset T_U(M_g, \bar{x}) \subset T_B(M_g, \bar{x}).$$

Let now $u \in T_B(M_g, \bar{x})$, meaning that there exist $(t_n) \downarrow 0, (u_n) \rightarrow u, n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,

$$g(\bar{x} + t_n u_n) \leq 0.$$

Since g is differentiable, there exists $(v_n) \rightarrow 0$ such that for all $n \geq n_0$

$$g(\bar{x} + t_n u_n) = g(\bar{x}) + t_n \nabla g(\bar{x})(u_n) + t_n v_n,$$

i.e.,

$$t_n (\nabla g(\bar{x})(u_n) + v_n) \leq 0.$$

Whence, passing to the limit in the relation $\nabla g(\bar{x})(u_n) + v_n \leq 0$, one gets that $\nabla g(\bar{x})(u) \leq 0$.

Take now $u \in X$ such that $\nabla g(\bar{x})(u) < 0$. Notice that such an element exists since $\nabla g(\bar{x}) \neq 0$. Take $(t_n) \downarrow 0$ and $(u_n) \rightarrow u$. Again, the differentiability property of g means

$$\begin{aligned} g(\bar{x} + t_n u_n) &= g(\bar{x}) + t_n \nabla g(\bar{x})(u_n) + t_n v_n \\ &= t_n (\nabla g(\bar{x})(u_n) + v_n), \end{aligned}$$

with $(v_n) \rightarrow 0$. Since $\nabla g(\bar{x})(u) < 0$ and $(u_n) \rightarrow u$, for all n large enough, $\nabla g(\bar{x})(u_n) + v_n < 0$, whence $g(\bar{x} + t_n u_n) < 0$. This means that $\bar{x} + t_n u_n \in M_g$ and we get $u \in T_{DM}(M_g, \bar{x})$.

Let now $v \in X$ such that $\nabla g(\bar{x})(v) \leq 0$ and $\lambda \in (0, 1)$. Clearly

$$\nabla g(\bar{x})(\lambda u + (1 - \lambda)v) < 0,$$

whence, from the previous step, $\lambda u + (1 - \lambda)v \in T_{DM}(M_g, \bar{x})$. Passing to the limit with $\lambda \rightarrow 0$, we get $v \in \text{cl } T_{DM}(M_g, \bar{x})$, and all the inclusions are proved.

These facts are well-known, but what we want to underline here is that the essential assumption for getting (2.2) is $\nabla g(\bar{x}) \neq 0$, which for a scalar function is equivalent to the Mangasarian-Fromowitz condition, that is, to exist $u \in X$ such that $\nabla g(\bar{x})(u) < 0$.

Now condition (2.1) becomes

$$\nabla f(\bar{x})(u) \geq 0, \text{ subject to } \nabla g(\bar{x})(u) \leq 0,$$

which can be seen as the fact that $0 \in X$ is an optimal solution to the linear problem

$$\min \nabla f(\bar{x})(u), \text{ subject to } \nabla g(\bar{x})(u) \leq 0.$$

Then, since for linear problems there is no need of supplementary qualification conditions for applying Karush-Kuhn-Tucker Theorem, we get $\lambda \geq 0$ such that

$$\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) = 0.$$

Suppose now that the constraint is expressed in the same way, but with a function $g = (g_1, g_2) : X \rightarrow \mathbb{R}^2$, where $g(x) \leq 0$ means that $g_1(x) \leq 0$ and $g_2(x) \leq 0$. Let \bar{x} be a feasible point. In the case of active constraints, that is, $g(\bar{x}) = 0 \in \mathbb{R}^2$, the Mangasarian-Fromowitz condition is: there exists $u \in X$ such that $\nabla g_1(\bar{x})(u) < 0$ and $\nabla g_2(\bar{x})(u) < 0$. On the same lines as before, this condition ensures that

$$\begin{aligned} T_B(M_g, \bar{x}) &= T_U(M_g, \bar{x}) = \text{cl } T_{DM}(M_g, \bar{x}) \\ &= \{u \in X \mid \nabla g_1(\bar{x})(u) \leq 0, \nabla g_2(\bar{x})(u) \leq 0\}. \end{aligned}$$

In particular, this means that

$$T_B(M_g, \bar{x}) = T_B(M_{g_1}, \bar{x}) \cap T_B(M_{g_2}, \bar{x})$$

and, in fact, this is the essential relationship to get, on the same argument as in the case of a single scalar-valued constraint, that there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0.$$

Now if another scalar-valued constraint is coming into play, that is, if we have $g = (g_1, g_2, g_3) : X \rightarrow \mathbb{R}^3$, then for an optimal solution \bar{x} , if the new constraint is active as well, the corresponding Mangasarian-Fromowitz condition (there exists $u \in X$ such that $\nabla g_i(\bar{x})(u) < 0$ for $i \in \overline{1, 3}$) ensures, similarly,

$$T_B(M_g, \bar{x}) = \bigcap_{i=1}^3 T_B(M_{g_i}, \bar{x}),$$

and then the existence of $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^3 \lambda_i \nabla g_i(\bar{x}) = 0.$$

Therefore, according to the facts above, every time we consider another scalar-valued constraint which is active at the underlying point \bar{x} , one has to impose the Mangasarian-Fromowitz condition, and this condition is stronger than the individual Mangasarian-Fromowitz conditions for every of the components of g .

3 A metric condition

Our aim is to present a situation when one can replace general Mangasarian-Fromowitz condition with individual Mangasarian-Fromowitz conditions under a supplementary hypothesis. This idea will be subject of some generalizations, since it will be clear that the additional assumption we impose can be extended to nondifferentiable settings.

Consider again the situation of $g = (g_1, g_2) : X \rightarrow \mathbb{R}^2$. Suppose that $g_1(\bar{x}) = 0$, $g_2(\bar{x}) = 0$, $\nabla g_1(\bar{x}) \neq 0$ and $\nabla g_2(\bar{x}) \neq 0$. Firstly, looking again at the above arguments, observe that we have

$$\begin{aligned} T_B(M_g, \bar{x}) &\subset T_B(M_{g_1}, \bar{x}) \cap T_B(M_{g_2}, \bar{x}) = T_U(M_{g_1}, \bar{x}) \cap T_B(M_{g_2}, \bar{x}) \\ &= T_B(M_{g_1}, \bar{x}) \cap T_U(M_{g_2}, \bar{x}) = T_U(M_{g_1}, \bar{x}) \cap T_U(M_{g_2}, \bar{x}). \end{aligned}$$

In other words, the general Mangasarian-Fromowitz condition is used exactly to show that we have the reverse inclusion in the above relation. However, such an inclusion can be obtained as well via some regularity assumptions on the sets. We refer the reader to the paper [4] and the references therein for some steps in this direction of investigation.

Theorem 3.1 *Let X be a normed vector space and $M_1, M_2 \subset X$ be closed sets. Take $\bar{x} \in M_1 \cap M_2$. Suppose that the following regularity assumption holds: there exist $s > 0, \mu > 0$ such that for all $x \in B(\bar{x}, s) \cap M_1$,*

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2). \quad (3.1)$$

Then

$$\begin{aligned} T_B(M_1, \bar{x}) \cap T_U(M_2, \bar{x}) &\subset T_B(M_1 \cap M_2, \bar{x}) \\ T_U(M_1, \bar{x}) \cap T_B(M_2, \bar{x}) &\subset T_B(M_1 \cap M_2, \bar{x}) \\ T_U(M_1, \bar{x}) \cap T_U(M_2, \bar{x}) &= T_U(M_1 \cap M_2, \bar{x}). \end{aligned}$$

Proof. Take $u \in T_B(M_1, \bar{x}) \cap T_U(M_2, \bar{x})$, i.e., $u \in T_B(M_1, \bar{x})$ and $u \in T_U(M_2, \bar{x})$. Then there exist $(t_n) \downarrow 0, (u_n) \rightarrow u, (v_n) \rightarrow u$ with $\bar{x} + t_n u_n \in M_1$ and $\bar{x} + t_n v_n \in M_2$ for all n large enough. Then one can apply the regularity assumption since $\bar{x} + t_n u_n \in B(\bar{x}, s)$ for n large enough: there exists $p_n \in M_1 \cap M_2$ with

$$\|\bar{x} + t_n u_n - p_n\| < \mu \|\bar{x} + t_n u_n - \bar{x} - t_n v_n\| + t_n^2.$$

Then for every n as above,

$$\|\bar{x} + t_n u_n - p_n\| < \mu \cdot t_n \|u_n - v_n\| + t_n^2,$$

whence

$$\|t_n^{-1}(p_n - \bar{x}) - u_n\| < \mu \|u_n - v_n\| + t_n.$$

We infer that $u'_n := t_n^{-1}(p_n - \bar{x}) \rightarrow u$ which, by the fact that for all n large enough

$$\bar{x} + t_n u'_n = p_n \in M_1 \cap M_2,$$

allows us to conclude the proof of the first inclusion of the theorem. Now, the other relations are similar. Notice that for the equality in the third relation one takes into account the simple inclusion

$$T_U(M_1 \cap M_2, \bar{x}) \subset T_U(M_1, \bar{x}) \cap T_U(M_2, \bar{x}).$$

The proof is complete. □

Remark 3.2 Observe that if X is finite dimensional, under condition (3.1) one can prove the following stronger assertion: for all $u \in T_B(M_1, \bar{x})$ and $v \in T_U(M_2, \bar{x})$, there is $w \in T_B(M_1 \cap M_2, \bar{x})$ such that

$$\|w - u\| \leq \mu \|v - u\|.$$

Indeed, one can follow with obvious modifications the proof above and observe that the sequence $(t_n^{-1}(p_n - \bar{x}))$ is bounded, whence it has a convergent subsequence whose limit satisfies the requirements for w .

Let us comment on the metric condition (3.1). A well-known and intensively studied regularity property for sets is the so-called metric inequality (see [10], [11], [15], [14], and the references therein): there exist $s > 0, \mu > 0$ such that for all $x \in B(\bar{x}, s)$,

$$d(x, M_1 \cap M_2) \leq \mu (d(x, M_1) + d(x, M_2)). \quad (3.2)$$

Moreover, in the mentioned paper [4] we used some metric subregularity conditions in getting calculus rules for tangent cones. We recall that a function $f : X \rightarrow Y$ is metrically subregular at $(\bar{x}, f(\bar{x}))$ with respect to $M \subset X$ when $\bar{x} \in M$ and there exist $s > 0, \mu > 0$ such that for every $u \in B(\bar{x}, s) \cap M$

$$d(u, f^{-1}(f(\bar{x})) \cap M) \leq \mu \|f(\bar{x}) - f(u)\|.$$

Now we prove the equivalence (up to a change of the involved constants) of all these conditions.

Proposition 3.3 Take $\bar{x} \in M_1 \cap M_2$. The next assertions are equivalent:

(i) there exist $s, \mu > 0$ such that for all $x \in B(\bar{x}, s) \cap M_1$,

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2).$$

(ii) there exist $r, t, \mu > 0$ such that for all $x \in B(\bar{x}, r) \cap M_1$,

$$d(x, M_1 \cap M_2) \leq \mu d(x, B(\bar{x}, t) \cap M_2).$$

(iii) there exist $r, \nu > 0$ such that for all $x \in B(\bar{x}, r) \cap M_2$,

$$d(x, M_1 \cap M_2) \leq \nu d(x, M_1).$$

(iv) there exist $r, \nu > 0$ such that for all $x \in B(\bar{x}, r)$,

$$d(x, M_1 \cap M_2) \leq \nu (d(x, M_1) + d(x, M_2)).$$

(v) the function $g : X \times X \rightarrow X$ given by $g(x, y) := x - y$ is metrically subregular at $(\bar{x}, \bar{x}, 0)$ with respect to $M_1 \times M_2$.

(vi) the function $h : X \times X \rightarrow \mathbb{R}$ given by $h(x, y) := d(x, y)$ is metrically subregular at $(\bar{x}, \bar{x}, 0)$ with respect to $M_1 \times M_2$.

Proof. (i) \Rightarrow (ii) This is obvious for $r := s$, since $d(x, M_2) \leq d(x, B(\bar{x}, t) \cap M_2)$.

(ii) \Rightarrow (i) Consider $s := \min\{r, t\}$ and $x \in B(\bar{x}, 3^{-1}s) \cap M_1$. Then for every $\varepsilon \in (0, 3^{-1}s)$, there is $x_\varepsilon \in M_2$ such that

$$\|x - x_\varepsilon\| < d(x, M_2) + \varepsilon \leq \|x - \bar{x}\| + \varepsilon < 2 \cdot 3^{-1}s.$$

Then

$$\|x_\varepsilon - \bar{x}\| \leq \|x - x_\varepsilon\| + \|x - \bar{x}\| < s \leq t,$$

whence $x_\varepsilon \in B(\bar{x}, t) \cap M_2$, so

$$d(x, M_1 \cap M_2) \leq \mu d(x, B(\bar{x}, t) \cap M_2) \leq \mu \|x - x_\varepsilon\| < \mu (d(x, M_2) + \varepsilon).$$

Since this is true for every ε small enough, we arrive at the conclusion.

(i) \Rightarrow (iv) Define $r = 2^{-1}s$ and take $x \in B(\bar{x}, r)$. Then $d(x, M_1) \leq \|x - \bar{x}\| < 2^{-1}s$, hence there is $m_1 \in M_1$ such that $\|x - m_1\| < 2^{-1}s$. Therefore,

$$\|m_1 - \bar{x}\| \leq \|m_1 - x\| + \|x - \bar{x}\| < s,$$

$m_1 \in M_1 \cap B(\bar{x}, s)$, and by (i)

$$d(m_1, M_1 \cap M_2) \leq \mu d(m_1, M_2).$$

But then

$$\begin{aligned} d(x, M_1 \cap M_2) &\leq \|x - m_1\| + d(m_1, M_1 \cap M_2) \\ &\leq \|x - m_1\| + \mu d(m_1, M_2) \\ &\leq \|x - m_1\| + \mu (\|m_1 - x\| + d(x, M_2)) \\ &\leq (1 + \mu) (\|x - m_1\| + d(x, M_2)). \end{aligned}$$

Since m_1 can be chosen such that $\|x - m_1\|$ is arbitrarily close to $d(x, M_1)$, it follows that (3.2) holds for $\nu := 1 + \mu$.

(iv) \Rightarrow (i) This is obvious for $s := r$ and $\mu := \nu$.

(iv) \Leftrightarrow (iii) This follows by the symmetry of conditions in the items (i) and (iii) and by (i) \Leftrightarrow (iv).

(i) \Rightarrow (v) We work on $X \times X$ with the box norm, but any equivalent norm can be used, by adjusting the involved constants accordingly. For $x \in M_1 \cap B(\bar{x}, s)$, we have by (i) that

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2).$$

Then for any $\varepsilon > 0$, there is $u_\varepsilon \in M_1 \cap M_2$ such that

$$\|x - u_\varepsilon\| \leq \mu d(x, M_2) + \varepsilon.$$

Moreover, for any $y \in B(\bar{x}, s) \cap M_2$,

$$\begin{aligned} \|y - u_\varepsilon\| &\leq \|y - x\| + \|x - u_\varepsilon\| \leq (1 + \mu) \|x - y\| + \varepsilon, \\ \|x - u_\varepsilon\| &\leq \mu \|x - y\| + \varepsilon, \end{aligned}$$

hence

$$\begin{aligned} \max \{\|x - u_\varepsilon\|, \|y - u_\varepsilon\|\} &\leq (1 + \mu) \|x - y\| + \varepsilon, \\ d((x, y), g^{-1}(0) \cap (M_1 \times M_2)) &= \inf_{u \in M_1 \cap M_2} \max \{\|x - u\|, \|y - u\|\} \leq (1 + \mu) \|x - y\| + \varepsilon. \end{aligned}$$

We may let $\varepsilon \rightarrow 0$ and we obtain that for any $(x, y) \in (B(\bar{x}, s) \times B(\bar{x}, s)) \cap (M_1 \times M_2)$,

$$\begin{aligned} d((x, y), g^{-1}(0) \cap (M_1 \times M_2)) &\leq (1 + \mu) \|x - y\| = (1 + \mu) d(0, g(x, y)) \\ &= (1 + \mu) \|g(\bar{x}, \bar{x}) - g(x, y)\|, \end{aligned}$$

hence (v) holds.

(v) \Rightarrow (ii) Suppose that there are $s, \mu > 0$ such that for any $(x, y) \in (B(\bar{x}, s) \times B(\bar{x}, s)) \cap (M_1 \times M_2)$,

$$d((x, y), g^{-1}(0) \cap (M_1 \times M_2)) \leq \mu d(0, g(x, y)).$$

Then, for $x \in M_1 \cap B(\bar{x}, s)$, and for any $y \in M_2 \cap B(\bar{x}, s)$,

$$d(x, M_1 \cap M_2) = \inf_{u \in M_1 \cap M_2} \|x - u\| \leq \inf_{u \in M_1 \cap M_2} \max \{\|x - u\|, \|y - u\|\} \leq \mu \|x - y\|.$$

In conclusion, for any $y \in M_2 \cap B(\bar{x}, s)$,

$$d(x, M_1 \cap M_2) \leq \mu \|x - y\|,$$

hence

$$d(x, M_1 \cap M_2) \leq \mu d(x, M_2 \cap B(\bar{x}, s)).$$

(v) \Leftrightarrow (vi) Observe that $g^{-1}(0) = h^{-1}(0)$ and that $d(0, g(x, y)) = \|x - y\| = d(x, y) = d(0, d(x, y)) = d(0, h(x, y))$. \square

Remark 3.4 Notice that (3.1) is exactly the fact that \bar{x} is a local minimum on M_1 of the function

$$x \mapsto \mu d_{M_2}(x) - d_{M_1 \cap M_2}(x).$$

Since the latter function is $(1 + \mu)$ -Lipschitz, by Clarke's penalization principle ([3, Proposition 2.4.3]), \bar{x} is a local minimum (without constraints) of

$$x \mapsto \mu d_{M_2}(x) - d_{M_1 \cap M_2}(x) + (1 + \mu) d_{M_1}(x),$$

which implies that for x around \bar{x}

$$d_{M_1 \cap M_2}(x) \leq \mu d_{M_2}(x) + (1 + \mu) d_{M_1}(x) \leq (1 + \mu) (d_{M_1}(x) + d_{M_2}(x)).$$

This is another argument for (i) \Rightarrow (iv) of the proposition above.

Remark 3.5 The fact that \bar{x} is a local minimum on M_1 of the function

$$x \mapsto \mu d_{M_2}(x) + \iota_{M_1} - d_{M_1 \cap M_2}(x), \tag{3.3}$$

where ι is the indicator function, means that the condition (3.1) can be seen as a minimality condition for a difference function. Necessary and sufficient conditions for it can be devised in terms of subdifferentials of the involved functions (hence, in our case, in terms of normal cones of $M_1, M_2, M_1 \cap M_2$) following the results in [16]. Moreover, in the convex case (that is, both M_1 and M_2 are convex) the local minimality becomes global minimality, and an equivalent condition in terms of ε -subdifferentials (and ε -normal cones) is to be found in [9].

For instance, on Asplund spaces, a well-known consequence of relation (3.2) is a formula for the Fréchet normal cone of the intersection of sets. But this follows as well when we apply the generalized Fermat rule to the function (3.3) to get that

$$0 \in \widehat{\partial}(\mu d_{M_2} + \iota_{M_1} - d_{M_1 \cap M_2})(\bar{x}).$$

Moreover, since the functions $\mu d_{M_2} + \iota_{M_1}$ and $d_{M_1 \cap M_2}$ are finite at \bar{x} , and $\widehat{\partial}d_{M_1 \cap M_2}(\bar{x}) = \widehat{N}(M_1 \cap M_2, \bar{x}) \cap D(0, 1) \neq \emptyset$, it follows by the difference calculus rule for the Fréchet subdifferential (1.1) that

$$0 \in \bigcap_{x^* \in \widehat{N}(M_1 \cap M_2, \bar{x}) \cap D(0, 1)} \left[\widehat{\partial}(\mu d_{M_2} + \iota_{M_1})(\bar{x}) - x^* \right].$$

This means that

$$\widehat{N}(M_1 \cap M_2, \bar{x}) \cap D(0, 1) \subset \widehat{\partial}(\mu d_{M_2} + \iota_{M_1})(\bar{x})$$

and, since ι_{M_1} is lower semicontinuous and μd_{M_2} is Lipschitz, one applies the approximate calculus rule for the Fréchet subdifferential on Asplund spaces to get $x_\varepsilon \in M_1 \cap B(\bar{x}, \varepsilon)$ and $y_\varepsilon \in M_2 \cap B(\bar{x}, \varepsilon)$ such that

$$\widehat{\partial}(\mu d_{M_2} + \iota_{M_1})(\bar{x}) \subset \widehat{\partial}(\mu d_{M_2})(x_\varepsilon) + \widehat{\partial}(\iota_{M_1})(y_\varepsilon) + \varepsilon D(0, 1).$$

Then,

$$\widehat{N}(M_1 \cap M_2, \bar{x}) \cap D(0, 1) \subset \widehat{N}(M_2, x_\varepsilon) \cap D(0, \mu) + \widehat{N}(M_1, y_\varepsilon) + \varepsilon D(0, 1).$$

In particular, by the fact that the involved objects are cones, by adjusting the involved constants if needed, it follows that for any $\varepsilon > 0$, there exist $x_\varepsilon \in M_1 \cap B(\bar{x}, \varepsilon)$ and $y_\varepsilon \in M_2 \cap B(\bar{x}, \varepsilon)$ such that

$$\widehat{N}(M_1 \cap M_2, \bar{x}) \subset \widehat{N}(M_1, x_\varepsilon) + \widehat{N}(M_2, y_\varepsilon) + \varepsilon D(0, 1).$$

Remark 3.6 Notice that, for the convex case, the necessity of some metric inequalities for ensuring the validity of constraints qualification conditions is discussed in [17].

Combining the discussions above, we present two consequences for the systems with multiple constraints considered in the previous section.

Corollary 3.7 Let $g = (g_1, g_2) : X \rightarrow \mathbb{R}^2$ be differentiable, consider $\bar{x} \in X$ such that $g_1(\bar{x}) = g_2(\bar{x}) = 0$ and $\nabla g_1(\bar{x}) \neq 0, \nabla g_2(\bar{x}) \neq 0$. If there exist $s > 0, \mu > 0$ such that for all $x \in B(\bar{x}, s) \cap M_{g_1}$,

$$d(x, M_{g_1} \cap M_{g_2}) \leq \mu d(x, M_{g_2}),$$

then

$$T_B(M_g, \bar{x}) = \{u \in X \mid \nabla g(\bar{x})(u) \leq 0\}.$$

Corollary 3.8 Let $g = (g_1, g_2, g_3) : X \rightarrow \mathbb{R}^3$ be differentiable, consider $\bar{x} \in X$ such that $g_i(\bar{x}) = 0$ and $\nabla g_i(\bar{x}) \neq 0$ for $i \in \overline{1, 3}$. If there exist $s, t, \mu, \gamma > 0$ such that for all $x \in B(\bar{x}, s) \cap M_{g_1}$,

$$d(x, M_{g_1} \cap M_{g_2}) \leq \mu d(x, M_{g_2}),$$

and for all $x \in B(\bar{x}, t) \cap M_{g_1} \cap M_{g_2}$

$$d(x, M_{g_1} \cap M_{g_2} \cap M_{g_3}) \leq \gamma d(x, M_{g_3}),$$

then

$$T_B(M_g, \bar{x}) = \{u \in X \mid \nabla g(\bar{x})(u) \leq 0\}.$$

We present some illustrative examples.

Example 3.9 1. Let $g_1, g_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$g_1(x, y) = x - y, \quad g_2(x, y) = x^2 - x + y.$$

Consider the sets

$$M_1 = M_{g_1} = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \quad \text{and} \quad M_2 = M_{g_2} = \{(x, y) \in \mathbb{R}^2 \mid y \leq x - x^2\}.$$

Take $(\bar{x}, \bar{y}) = 0 \in \mathbb{R}^2$ and observe that $\nabla g_1(\bar{x}, \bar{y}) = (1, -1) = -\nabla g_2(\bar{x}, \bar{y})$ and Mangasarian-Fromowitz condition holds for these functions individually at (\bar{x}, \bar{y}) , while it does not for (g_1, g_2) . We have

$$T_B(M_1, 0) = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \quad \text{and} \quad T_B(M_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid y \leq x\},$$

whence

$$T_B(M_1, 0) \cap T_B(M_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid x = y\}.$$

On the other hand, $M_1 \cap M_2 = \{0\}$, so $T_B(M_1 \cap M_2, 0) = \{0\}$, whence the equality does not hold. However, the relation (3.1) does not hold either. Indeed, the relation (3.1) would imply the existence of $\mu > 0$ such that for every small $x > 0$,

$$x < x\sqrt{2} = d((x, x), M_1 \cap M_2) \leq \mu d((x, x), M_2) \leq \mu \|(x, x) - (x, x - x^2)\| = \mu x^2,$$

which is impossible.

2. Let $g_1, g_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$g_1(x, y) = x - y, \quad g_2(x, y) = -x + y.$$

Consider the sets

$$M_1 = M_{g_1} = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \\ M_2 = M_{g_2} = \{(x, y) \in \mathbb{R}^2 \mid y \leq x\}.$$

Take $(\bar{x}, \bar{y}) = 0 \in \mathbb{R}^2$ and observe that $\nabla g_1(\bar{x}) = (1, -1) = -\nabla g_2(\bar{x})$ and Mangasarian-Fromowitz condition holds for these functions individually at (\bar{x}, \bar{y}) , while it does not for (g_1, g_2) . We have

$$T_B(M_1, 0) = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \quad \text{and} \quad T_B(M_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid y \leq x\},$$

whence

$$T_B(M_1, 0) \cap T_B(M_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid x = y\}.$$

On the other hand,

$$M_1 \cap M_2 = \{(x, y) \in \mathbb{R}^2 \mid x = y\},$$

so

$$T_B(M_1 \cap M_2, 0) = \{(x, y) \in \mathbb{R}^2 \mid x = y\},$$

whence the equality does hold. Moreover, it is not difficult to see that the relation (3.1) does hold as well for small s and $\mu = 1$.

The last example above shows that the metric condition (3.1) is either strictly weaker than the global Mangasarian-Fromowitz condition, or independent from it. We do not know which of these two situations is actually true. The paper [1] contains an example of convex cones for which equality between tangent cones takes place, while the metric inequality does not, but this example is not fitted to our situation here.

Nevertheless, the condition (3.1) and the global Mangasarian-Fromowitz condition are only sufficient for having the equality between the tangent cone to the intersection and the intersection of tangent cones, as the next example shows.

Example 3.10 Consider $g_1, g_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$g_1(x, y) = \begin{cases} x^4 \sin^2 \frac{1}{x} - y, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$g_2(x, y) = \begin{cases} x^4 \sin^2 \frac{1}{x} + y, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

respectively. We have that $\nabla g_1(0, 0) = (0, -1)$ and $\nabla g_2(0, 0) = (0, 1)$ whence the individual Mangasarian-Fromowitz conditions hold, while the global one does not. We have that

$$\begin{aligned} T_B(M_{g_1}, (0, 0)) \cap T_B(M_{g_2}, (0, 0)) &= T_B(M_{g_1} \cap M_{g_2}, (0, 0)) \\ &= \{(a, 0) \mid a \in \mathbb{R}\}. \end{aligned}$$

Now denote $f(x) = x^4 \sin^2 \frac{1}{x}$ for any $x \neq 0$ and consider, for $n \in \mathbb{N} \setminus \{0\}$, the number

$$\bar{x}_n = \frac{1}{2\pi} \left(\frac{1}{n} + \frac{1}{n+1} \right) = \frac{2n+1}{2\pi n(n+1)},$$

and the element

$$(\bar{x}_n, f(\bar{x}_n)) \in M_{g_1}.$$

Since

$$M_{g_1} \cap M_{g_2} = \{(0, 0)\} \cup \left\{ \left(\frac{1}{k\pi}, 0 \right) \mid k \in \mathbb{Z} \setminus \{0\} \right\},$$

one has that

$$\left| \bar{x}_n - \frac{1}{n\pi} \right| \leq d((\bar{x}_n, f(\bar{x}_n)), M_{g_1} \cap M_{g_2}).$$

On the other hand,

$$d((\bar{x}_n, f(\bar{x}_n)), M_{g_2}) \leq 2f(\bar{x}_n),$$

whence, (3.1) would imply the existence of a constant $\mu > 0$ such that for all n large enough,

$$\left| \bar{x}_n - \frac{1}{n\pi} \right| = \frac{1}{2\pi n(n+1)} \leq 2\mu \left(\frac{2n+1}{2\pi n(n+1)} \right)^4 \sin^2 \left(\frac{2\pi n(n+1)}{2n+1} \right),$$

that is

$$4\pi^3 \leq \mu \frac{(2n+1)^4}{n^3(n+1)^3} \sin^2 \left(\frac{2\pi n(n+1)}{2n+1} \right).$$

Of course, this is impossible since the right-hand term converges to 0 as $n \rightarrow \infty$.

4 Consequences in optimization

In this section we present some consequences of the metric condition discussed above and we give as well some other similar conditions that can be used in various contexts.

Consider a scalar function $f : X \rightarrow \mathbb{R}$ and recall that the Hadamard upper directional derivative of f at $\bar{x} \in X$ in direction $u \in X$ is

$$d_+ f(\bar{x}, u) = \limsup_{u' \rightarrow u, t \downarrow 0} t^{-1}(f(\bar{x} + tu') - f(\bar{x})),$$

while the Hadamard lower directional derivative of f at \bar{x} in direction u is

$$d_- f(\bar{x}, u) = \liminf_{u' \rightarrow u, t \downarrow 0} t^{-1}(f(\bar{x} + tu') - f(\bar{x})).$$

In [2] a concept of directional minimum was introduced and studied in the vectorial setting (see also [7]). We consider this concept here, but, initially, for the sake of simplicity, we restrict ourselves to the scalar case.

Definition 4.1 *Let $f : X \rightarrow \mathbb{R}$ be a function and $A \subset X$, $L \subset S(0,1)$ be nonempty closed sets. One says that $\bar{x} \in A$ is a local directional minimum point for f on A with respect to (the set of directions) L if there exists a neighborhood U of \bar{x} such that for every $x \in U \cap A \cap (\bar{x} + \text{cone } L)$, $f(\bar{x}) \leq f(x)$.*

Proposition 4.2 *Let $f : X \rightarrow \mathbb{R}$ be a function, $A \subset X$, $L \subset S(0,1)$ be nonempty closed sets and $\bar{x} \in A$. Suppose that there exist $s > 0, \mu > 0$ such that*

$$\forall x \in B(\bar{x}, s) \cap A : d(x, A \cap (\bar{x} + \text{cone } L)) \leq \mu d(x, \bar{x} + \text{cone } L). \quad (4.1)$$

(i) *If \bar{x} is a local directional minimum point for f on A with respect to L , then $d_+ f(\bar{x}, u) \geq 0$ for all $u \in T_B(A, \bar{x}) \cap L$.*

(ii) *Moreover, if X is finite dimensional and $d_- f(\bar{x}, u) > 0$ for all $u \in T_B(A, \bar{x}) \cap L$, then there exists $\alpha > 0$ such that \bar{x} is a local directional minimum point for $f(\cdot) + \alpha \|\cdot - \bar{x}\|$ on A with respect to L .*

Proof. Firstly, observe that, according to Theorem 3.1, the metric assumption imposed ensures that

$$T_B(A, \bar{x}) \cap T_U(\bar{x} + \text{cone } L, \bar{x}) \subset T_B(A \cap (\bar{x} + \text{cone } L), \bar{x}).$$

Since, obviously,

$$T_B(A \cap (\bar{x} + \text{cone } L), \bar{x}) \subset T_B(A, \bar{x}) \cap T_B(\bar{x} + \text{cone } L, \bar{x})$$

and

$$T_B(\bar{x} + \text{cone } L, \bar{x}) = T_U(\bar{x} + \text{cone } L, \bar{x}) = T_U(\text{cone } L, 0) = \text{cone } L,$$

we actually get

$$T_B(A \cap (\bar{x} + \text{cone } L), \bar{x}) = T_B(A, \bar{x}) \cap \text{cone } L.$$

Now the proof of both items is quite standard, but we present it here for the sake of clarity.

(i) Remark that the minimality of \bar{x} on A with respect to a set of directions L is in fact the minimality of \bar{x} on $A \cap (\bar{x} + \text{cone } L)$. Therefore, for every $u \in T_B(A, \bar{x}) \cap \text{cone } L = T_B(A \cap$

$(\bar{x} + \text{cone } L), \bar{x})$, there are some sequences $(t_n) \downarrow 0, (u_n) \rightarrow u$ such that for all $n \in \mathbb{N}, \bar{x} + t_n u_n \in U \cap A \cap (\bar{x} + \text{cone } L)$, whence, for n large enough $f(\bar{x} + t_n u_n) \geq f(\bar{x})$. Consequently,

$$d_+ f(\bar{x}, u) \geq \limsup_{n \rightarrow \infty} t_n^{-1} (f(\bar{x} + t_n u_n) - f(\bar{x})) \geq 0.$$

(ii) Suppose, by way of contradiction, that for every positive integer n there exists $x_n \in A \cap (\bar{x} + \text{cone } L) \cap B(\bar{x}, n^{-1})$ such that

$$f(x_n) < f(\bar{x}) - n^{-1} \|x_n - \bar{x}\|.$$

Then for all $n \geq 1, x_n \neq \bar{x}$ and

$$\|x_n - \bar{x}\|^{-1} (f(x_n) - f(\bar{x})) = \|x_n - \bar{x}\|^{-1} \left(f\left(\bar{x} + \|x_n - \bar{x}\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\right) - f(\bar{x}) \right) < -\frac{1}{n}.$$

Since X is finite dimensional, we can suppose, without loss of generality, that $\left(\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\right)$ converges to an element u which clearly is in $T_B(A, \bar{x}) \cap L$. Therefore

$$d_- f(\bar{x}, u) \leq \liminf_{n \rightarrow \infty} \|x_n - \bar{x}\|^{-1} \left(f\left(\bar{x} + \|x_n - \bar{x}\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\right) - f(\bar{x}) \right) \leq 0,$$

and this is a contradiction. Therefore, the conclusion holds. \square

Remark 4.3 According to Proposition 3.3, one can replace (4.1) by any of the equivalent corresponding conditions.

It is interesting, from our point of view, the fact that metric conditions of the type (3.1) come into play as weak assumptions to ensure the validity of some penalization principles. Even if all the conditions in Proposition 3.3 are equivalent, the change of constants is important in the construction of the penalty function. More details will be given after the next results.

Theorem 4.4 (penalty of an intersection of sets) Let $f : X \rightarrow \mathbb{R}$ be a function and $A, B \subset X$ be nonempty, closed sets. Let $\bar{x} \in A \cap B$ be a local minimum point for f on $A \cap B$. Suppose that

- (i) there exist $\varepsilon > 0$ and $\ell > 0$ such that f is ℓ -Lipschitz on $B(\bar{x}, \varepsilon)$;
- (ii) there exist $s > 0, \mu > 0$ such that for all $x \in B(\bar{x}, s) \cap A$,

$$d(x, A \cap B) \leq \mu d(x, B).$$

Then \bar{x} is a local minimum point for $f + \ell \mu d(\cdot, B)$ on A and a local minimum point (without constraints) for

$$f + \ell(1 + \mu) d(\cdot, A) + \ell \mu d(\cdot, B).$$

Proof. We prove first that \bar{x} is a local minimum point on A for

$$f + \ell \mu d(\cdot, B).$$

Let $\alpha > 0$ be the radius of the ball given by the local minimality of \bar{x} . We choose $\delta > 0$ such that $\delta < \min \left\{ (1 + \mu)^{-1} \alpha, s, (1 + \mu)^{-1} \varepsilon \right\}$ and $x \in A \cap B(\bar{x}, \delta)$. Since $f(\bar{x}) + \ell \mu d(\bar{x}, B) = f(\bar{x})$, we have to show that

$$f(\bar{x}) \leq f(x) + \ell \mu d(x, B).$$

If $x \in A \cap B(\bar{x}, \delta) \cap B$, this is obvious, by the property of \bar{x} . Take $x \in A \cap B(\bar{x}, \delta) \setminus B$. By (ii),

$$d(x, A \cap B) \leq \mu d(x, B),$$

whence, for every $\rho \in (0, \min\{\alpha, \varepsilon\} - \delta(1 + \mu))$,

$$d(x, A \cap B) < \mu d(x, B) + \rho.$$

This means that there is $x_\rho \in A \cap B$ such that

$$\|x - x_\rho\| < \mu d(x, B) + \rho \leq \mu \|x - \bar{x}\| + \rho.$$

But,

$$\|x_\rho - \bar{x}\| \leq \|x_\rho - x\| + \|x - \bar{x}\| \leq \mu\delta + \rho + \delta = \delta(1 + \mu) + \rho < \min\{\alpha, \varepsilon\}.$$

Consequently, $x_\rho \in A \cap B \cap B(\bar{x}, \alpha)$, whence $f(\bar{x}) \leq f(x_\rho)$. Then, by (i),

$$f(\bar{x}) \leq f(x_\rho) \leq f(x) + \ell \|x - x_\rho\| \leq f(x) + \ell\mu d(x, B) + \ell\rho.$$

Letting $\rho \rightarrow 0$ we get the desired inequality, whence the claim is proved.

Now, observe that $f + \ell\mu d(\cdot, B)$ is locally Lipschitz around \bar{x} with the Lipschitz constant $\ell(1 + \mu)$. We apply the standard Clarke penalization principle and we get the conclusion. \square

Remark 4.5 *In fact, the above result is more general than the usual Clarke penalization principle which corresponds to the case $A = B$.*

We apply this generalized penalty result for getting necessary optimality conditions in the dual space for directional minima.

Theorem 4.6 *Let X be an Asplund space, $f : X \rightarrow \mathbb{R}$ be a function, $A \subset X$, $L \subset S(0, 1)$ be nonempty closed sets and $\bar{x} \in A$. Suppose that:*

- (i) \bar{x} is a local directional minimum point for f on A with respect to L ;
- (ii) there exist $\varepsilon > 0$ and $\ell > 0$ such that f is ℓ -Lipschitz on $B(\bar{x}, \varepsilon)$;
- (iii) there exist $s > 0, \mu > 0$ such that for all $x \in B(\bar{x}, s) \cap A$,

$$d(x, A \cap (\bar{x} + \text{cone } L)) \leq \mu d(x, \bar{x} + \text{cone } L).$$

Then there are $u^* \in N(A, \bar{x}), v^* \in N(\text{cone } L, 0)$ with $\|u^*\| \leq \ell(1 + \mu)$ and $\|v^*\| \leq \ell\mu$, such that

$$-u^* - v^* \in \partial f(\bar{x}).$$

Proof. Since \bar{x} is a local directional minimum point for f on A with respect to L , then it is a local minimum point for f on $A \cap (\bar{x} + \text{cone } L)$. Then, on the basis of (ii) and (iii) we can apply Theorem 4.4 to get that \bar{x} is a local minimum point for $f + \ell(1 + \mu)d(\cdot, A) + \ell\mu d(\cdot, \bar{x} + \text{cone } L)$. Therefore, since all the functions are locally Lipschitz around \bar{x} , one can apply the exact subdifferential calculus rules (see [12, Theorem 3.36]) to get

$$\begin{aligned} 0 &\in \partial(f + \ell(1 + \mu)d(\cdot, A) + \ell\mu d(\cdot, \bar{x} + \text{cone } L))(\bar{x}) \\ &\subset \partial f(\bar{x}) + \ell(1 + \mu)\partial d(\cdot, A)(\bar{x}) + \ell\mu\partial d(\cdot, \bar{x} + \text{cone } L)(\bar{x}). \end{aligned}$$

The conclusion follows. \square

Remark 4.7 *The importance of having constants as small as possible is apparent in the conclusion of Theorem 4.6 in the estimation of the norms of the generalized Lagrange multipliers u^* and v^* .*

Remark 4.8 *Observe that if cone L is convex, then $N(\text{cone } L, 0) = L^-$.*

We illustrate the above results by the following example.

Example 4.9 Let $X := \mathbb{R}^2$, $A := [0, 1]^2$, $L := S((0, 0), 1) \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq x \geq 0\}$. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = 2y - x$. Clearly, $(0, 0)$ is a directional minimum point for f on A with respect to L . Moreover, the assumptions of Theorem 4.6 are satisfied with $\ell = \sqrt{5}$ and $\mu = 1$. Then $\partial f(0, 0) = \{(-1, 2)\}$ and there is $u^* = (0, -1) \in N(A, (0, 0))$, $v^* = (1, -1) \in L^-$ satisfying the conclusion.

On the other hand, if we change only the set of directions to

$$L := S((0, 0), 1) \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq 3^{-1} \cdot x \geq 0\},$$

then one cannot find u^* , v^* to satisfy the conclusion, confirming that this time $(0, 0)$ is not a directional minimum point for f on A with respect to L .

In the next result we point out that a metric condition can be imposed as well for a functional constraint in order to get a penalty result. Let Z be a normed vector space, $g : X \rightarrow Z$ be a function and $Q \subset Z$ be a pointed closed convex cone. One defines the set-valued map $\tilde{g} : X \rightrightarrows Z$ given by $\tilde{g}(x) = g(x) + Q$ and one considers $g^{-1}(-Q) = \tilde{g}^{-1}(0)$ as the feasible set.

Theorem 4.10 (penalty for a functional constraint) *Let $\bar{x} \in g^{-1}(-Q)$ be a local minimum of f on $g^{-1}(-Q)$. Suppose that*

- (i) *there exist $\varepsilon > 0$ and $\ell > 0$ such that f is ℓ -Lipschitz on $B(\bar{x}, \varepsilon)$;*
- (ii) *there exist $s, \mu > 0$ such that for all $x \in g^{-1}(-Q + B(0, s)) \cap B(\bar{x}, s)$*

$$d(x, g^{-1}(-Q)) \leq \mu d(0, \tilde{g}(x) \cap B(0, s)).$$

Then $(\bar{x}, 0)$ is a local minimum for the function $(x, z) \mapsto f(x) + \ell\mu \|z\| + \ell(1 + \mu)d((x, z), \text{Gr } \tilde{g})$.

Proof. Denote $h : X \times Z \rightarrow \mathbb{R}$, $h(x, z) = f(x) + \ell\mu \|z\|$. Clearly, $h(\bar{x}, 0) = f(\bar{x})$ and we show first that $(\bar{x}, 0)$ is a local minimum on $\text{Gr } \tilde{g}$ for h . Thus, we have to show that there is $\delta > 0$ such that for all $(x, z) \in (B(\bar{x}, \delta) \times B(0, \delta)) \cap \text{Gr } \tilde{g}$,

$$f(\bar{x}) \leq f(x) + \ell\mu \|z\|.$$

Let α be the radius of the ball given by the local minimality of \bar{x} . Take $\delta > 0$ such that $\delta < \min \left\{ \varepsilon(1 + \mu)^{-1}, s, \alpha(1 + \mu)^{-1} \right\}$ and $(x, z) \in (B(\bar{x}, \delta) \times B(0, \delta)) \cap \text{Gr } \tilde{g}$. If $z \in -Q$, then by the property of \bar{x} ,

$$f(\bar{x}) \leq f(x) \leq f(x) + \ell\mu \|z\|.$$

Suppose that $z \notin -Q$. Then there is $q \in Q$ such that $z - q = g(x)$, hence $x \in B(\bar{x}, \delta) \cap g^{-1}(z - q) \subset B(\bar{x}, s) \cap g^{-1}(-Q + B(0, s))$. Now, by (ii),

$$d(x, g^{-1}(-Q)) \leq \mu d(0, \tilde{g}(x) \cap B(0, s)) \leq \mu \|z\|.$$

Then for every $\rho \in (0, \min \{\alpha, \varepsilon\} - \delta(1 + \mu))$, there is $x_\rho \in g^{-1}(-Q)$ such that

$$\|x - x_\rho\| < \mu \|z\| + \rho \leq \mu\delta + \rho.$$

Then

$$\|x_\rho - \bar{x}\| \leq \|x_\rho - x\| + \|x - \bar{x}\| \leq \mu\delta + \rho + \delta = \delta(1 + \mu) + \rho < \min \{\alpha, \varepsilon\}.$$

Therefore, by (i),

$$f(\bar{x}) \leq f(x_\rho) \leq f(x) + \ell \|x - x_\rho\| \leq f(x) + \ell\mu \|z\| + \ell\rho.$$

For $\rho \rightarrow 0$, we get the claim.

Finally, since h is locally Lipschitz around $(\bar{x}, 0)$ with constant $\ell(1 + \mu)$, we apply the standard Clarke penalization principle and we get the conclusion. \square

Theorem 4.11 *In the notation above, let X, Z be Asplund spaces and $\bar{x} \in g^{-1}(-Q)$ be a local minimum of f on $g^{-1}(-Q)$. Suppose that*

- (i) *there exist $\varepsilon > 0$ and $\ell > 0$ such that f is ℓ -Lipschitz on $B(\bar{x}, \varepsilon)$;*
- (ii) *there exist $s, \mu > 0$ such that for all $x \in g^{-1}(-Q + B(0, s)) \cap B(\bar{x}, s)$*

$$d(x, g^{-1}(-Q)) \leq \mu d(0, \tilde{g}(x) \cap B(0, s)).$$

Then

$$-\partial f(\bar{x}) \cap D(0, \ell(1 + \mu)) \cap D^* \tilde{g}(\bar{x}, 0)(Q^+ \cap D(0, \ell\mu)) \neq \emptyset.$$

Proof. Following Theorem 4.10, $(\bar{x}, 0)$ is a local minimum for the function $(x, z) \mapsto f(x) + \ell\mu \|z\| + \ell(1 + \mu) d((x, z), \text{Gr } \tilde{g})$, whence

$$(0, 0) \in \partial f(\bar{x}) \times \{0\} + \{0\} \times \ell\mu \partial \|\cdot\| (0) + \ell(1 + \mu) \partial d((\cdot, \cdot), \text{Gr } \tilde{g})(\bar{x}, 0).$$

Then there are $x^* \in \partial f(\bar{x})$, $z^* \in D(0, \ell\mu)$ such that

$$\begin{aligned} (-x^*, -z^*) &\in N(\text{Gr } \tilde{g}, (\bar{x}, 0)) \text{ and} \\ \|(x^*, z^*)\| &\leq \ell(1 + \mu). \end{aligned}$$

In particular, this means that

$$\begin{aligned} \|x^*\| &\leq \ell(1 + \mu) \text{ and} \\ -x^* &\in D^* \tilde{g}(\bar{x}, 0)(z^*). \end{aligned}$$

Taking into account the definition of the limiting normal cone, $(-x^*, -z^*)$ can be approached in the weak-star topology by some elements $(-x_n^*, -z_n^*)$ elements in the Fréchet normal cone to $\text{Gr } \tilde{g}$ at points close to $(\bar{x}, 0)$. According to [5, Lemma 3.2], the elements z_n^* are in Q^+ and since this set is weakly-star closed, $z^* \in Q^+$. Therefore, $z^* \in Q^+ \cap D(0, \ell\mu)$ and then we get the conclusion. \square

By combining both penalty principles above, we can get optimality conditions for directional minima under functional constraints.

Corollary 4.12 *In the notation above, let \bar{x} be a local directional minimum point for f on $g^{-1}(-Q)$ with respect to L . Suppose that*

- (i) *there exist $\varepsilon > 0$ and $\ell > 0$ such that f is ℓ -Lipschitz on $B(\bar{x}, \varepsilon)$;*
- (ii) *there exist $s, \mu > 0$ such that for all $x \in g^{-1}(-Q + B(0, s)) \cap B(\bar{x}, s)$*

$$d(x, g^{-1}(-Q)) \leq \mu d(0, \tilde{g}(x) \cap B(0, s))$$

and for all $x \in B(\bar{x}, s) \cap g^{-1}(-Q)$,

$$d(x, g^{-1}(-Q) \cap (\bar{x} + \text{cone } L)) \leq \mu d(x, \bar{x} + \text{cone } L).$$

Then

$$-\partial f(\bar{x}) \cap D(0, \ell\mu + \ell(1 + \mu)^2) \cap [N(\text{cone } L, 0) \cap D(0, \ell\mu) + D^* \tilde{g}(\bar{x}, 0)(Q^+ \cap D(0, \ell(1 + \mu)\mu))] \neq \emptyset.$$

Proof. Using the assumptions, by Theorems 4.4 and 4.10, successively, \bar{x} is a minimum point for

$$f + \ell\mu d(\cdot, (\bar{x} + \text{cone } L))$$

on $g^{-1}(-Q)$ and $(\bar{x}, 0)$ is unconstrained minimum for

$$(x, z) \mapsto f(x) + \ell\mu d(x, (\bar{x} + \text{cone } L)) + \ell(1 + \mu)\mu \|z\| + \ell(1 + \mu)^2 d((x, z), \text{Gr } \tilde{g}).$$

Employing again the limiting calculus rule, we get that there are $x^* \in \partial f(\bar{x})$, $u^* \in N(\text{cone } L, 0)$, $z^* \in Z^*$ with $\|u^*\| \leq \ell\mu$, $\|z^*\| \leq \ell(1 + \mu)\mu$, $\|x^*\| \leq \ell\mu + \ell(1 + \mu)^2$ and

$$(-x^* - u^*, -z^*) \in N(\text{Gr } \tilde{g}, (\bar{x}, 0)).$$

As above, $z^* \in Q^+$ and summing up we have the conclusion. \square

Finally, we give some optimality conditions for a concept of directional minimum with respect to two sets of directions for vectorial functions (see [7]). One needs an auxiliary result.

Lemma 4.13 *Let X, Y be Banach spaces, $f : X \rightarrow Y$ a continuously differentiable function, $M \subset S_Y$ a closed and nonempty set, and $\bar{x} \in X$. Suppose that one of the following sets of conditions holds:*

- (i) *cone M is convex and there exists $u_0 \in X$ such that $\nabla f(\bar{x})(u_0) \in -\text{int}(\text{cone } M)$;*
- (ii) *the map $g : X \times Y \rightarrow Y$ given by $g(x, y) = f(x) - y$ is metrically subregular at $(\bar{x}, f(\bar{x}), 0)$ with respect to $X \times f^{-1}(f(\bar{x}) - \text{cone } M)$.*

Then

$$T_B(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x}) = \nabla f(\bar{x})^{-1}(-\text{cone } M). \quad (4.2)$$

Proof. Provided that f is continuously differentiable, it is always true that $T_B(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x}) \subset \nabla f(\bar{x})^{-1}(-\text{cone } M)$. Indeed, for an arbitrary $u \in T_B(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x})$, there exist $(t_n) \downarrow 0$ and $(u_n) \rightarrow u$ such that $f(\bar{x} + t_n u_n) \in f(\bar{x}) - \text{cone } M$ for all n . Since f is continuously differentiable at \bar{x} , this means that there exists a function $\alpha : X \rightarrow Y$ such that $\lim_{h \rightarrow 0} \alpha(\bar{x} + h) = \alpha(\bar{x}) = 0$ and

$$f(\bar{x} + t_n u_n) = f(\bar{x}) + t_n \nabla f(\bar{x})(u_n) + t_n \alpha(\bar{x} + t_n u_n) \|u_n\|.$$

Therefore,

$$t_n \nabla f(\bar{x})(u_n) + t_n \alpha(\bar{x} + t_n u_n) \|u_n\| \in -\text{cone } M,$$

and from here, dividing by t_n and passing to the limit, one gets $u \in \nabla f(\bar{x})^{-1}(-\text{cone } M)$.

If the condition from (ii) holds, the conclusion is granted by [4, Theorem 3.1].

Suppose now that we are under the assumptions from (i). Let (t_n) be an arbitrary sequence of positive numbers converging to 0, and set $u_n := u_0$. Using the differentiability of f , we can write

$$\begin{aligned} f(\bar{x} + t_n u_n) &= f(\bar{x}) + t_n \nabla f(\bar{x})(u_n) + t_n \alpha(\bar{x} + t_n u_n) \|u_n\| \\ &= f(\bar{x}) + t_n [\nabla f(\bar{x})(u_0) + \alpha(\bar{x} + t_n u_0) \|u_0\|], \end{aligned}$$

where $\alpha : X \rightarrow Y$ is such that $\lim_{n \rightarrow \infty} \alpha(\bar{x} + t_n u_n) = \alpha(\bar{x}) = 0$. Therefore, since $\nabla f(\bar{x})(u_0) \in -\text{int}(\text{cone } M)$, we have that for all n large enough, $f(\bar{x} + t_n u_0) \in f(\bar{x}) - \text{cone } M$, that is $u_0 \in T_U(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x})$.

Consider now $v \in X$ such that $\nabla f(\bar{x})(v) \in -\text{cone } M$, and $\lambda \in (0, 1)$. Then

$$\nabla f(\bar{x})(\lambda u_0 + (1 - \lambda)v) \in -(\text{int}(\text{cone } M) + \text{cone } M) = -\text{int}(\text{cone } M).$$

Hence, from the previous step,

$$\lambda u_0 + (1 - \lambda)v \in T_U(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x}).$$

Passing to the limit with $\lambda \rightarrow 0$, and taking into account that the adjacent cone is closed, we get that $v \in T_U(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x}) \subset T_B(f^{-1}(f(\bar{x}) - \text{cone } M), \bar{x})$, and this completes the proof. \square

We recall now the Gerstewitz (Tammer) scalarization functional (see, for instance [8, Section 2.3]).

Lemma 4.14 *Let $K \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \text{int } K$, the functional $s_e : Y \rightarrow \mathbb{R}$ given by*

$$s_e(y) = \inf\{t \in \mathbb{R} \mid te \in y + K\}$$

is continuous and sublinear. Moreover, for every $t \in \mathbb{R}$,

$$\{y \in Y \mid s_e(y) < t\} = te - \text{int } K.$$

We give the definition of efficiency from [7].

Definition 4.15 *Let X and Y be normed vector spaces, $K \subset Y$ a closed ordering cone with nonempty interior, $f : X \rightarrow Y$, and $A \subset X$, $L \subset S_X$, $M \subset S_Y$ closed sets. One says that $\bar{x} \in A$ is a weak local directional Pareto minimum point for f with respect to the sets of directions L and M on A if there exists a neighborhood U for \bar{x} such that*

$$[[f(A \cap U \cap (\bar{x} + \text{cone } L)) \cap (f(\bar{x}) - \text{cone } M)] - f(\bar{x})] \cap (-\text{int } K) = \emptyset.$$

We are now able to present the necessary optimality conditions.

Theorem 4.16 *Let X, Y be Banach spaces, $f : X \rightarrow Y$ a continuously differentiable function, $K \subset Y$ a closed convex ordering cone with nonempty interior, $A \subset X$, $L \subset S_X$ and $M \subset S_Y$ closed and nonempty sets, and $\bar{x} \in A$. Assume there exist $s, \mu, t, \gamma > 0$ such that*

$$d(x, A \cap (\bar{x} + \text{cone } L)) \leq \mu d(x, \bar{x} + \text{cone } L), \quad \forall x \in B(\bar{x}, s) \cap A,$$

and

$$d(x, A \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M)) \leq \gamma d(x, f^{-1}(f(\bar{x}) - \text{cone}M)),$$

$$\forall x \in B(\bar{x}, t) \cap A \cap (\bar{x} + \text{cone}L).$$

Suppose also that either (i) or (ii) from Lemma 4.13 hold, and \bar{x} is a weak local directional Pareto minimum for f on A with respect to L and M .

Then

$$\nabla f(\bar{x})(u) \notin -\text{int}K, \quad \forall u \in T_B(A, \bar{x}) \cap \text{cone}L \cap \nabla f(\bar{x})^{-1}(-\text{cone}M).$$

Proof. First, we observe that the metric assumptions ensure the equality

$$T_B(A \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M), \bar{x}) = T_B(A, \bar{x}) \cap T_B(\bar{x} + \text{cone}L, \bar{x}) \cap T_B(f^{-1}(f(\bar{x}) - \text{cone}M), \bar{x}).$$

But $T_B(\bar{x} + \text{cone}L, \bar{x}) = \text{cone}L$ and since we are under assumptions (i) or (ii) from Lemma 4.13, we also have

$$T_B(f^{-1}(f(\bar{x}) - \text{cone}M), \bar{x}) = \nabla f(\bar{x})^{-1}(-\text{cone}M).$$

Therefore,

$$T_B(A \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M), \bar{x}) = T_B(A, \bar{x}) \cap \text{cone}L \cap \nabla f(\bar{x})^{-1}(-\text{cone}M). \quad (4.3)$$

The minimality of \bar{x} means there exists a neighborhood U for \bar{x} such that

$$[[f(A \cap U \cap (\bar{x} + \text{cone}L)) \cap (f(\bar{x}) - \text{cone}M)] - f(\bar{x})] \cap (-\text{int}K) = \emptyset.$$

That is, for every $e \in \text{int}K$ and every $x \in A \cap U \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M)$, we have $s_e(f(x) - f(\bar{x})) \geq 0$ since $f(x) - f(\bar{x}) \notin -\text{int}K$. We can now see \bar{x} as a local minimum point for the scalar function $g : X \rightarrow \mathbb{R}$ given by $g(x) = s_e(f(x) - f(\bar{x}))$ on $A \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M)$. Note that $g(\bar{x}) = s_e(0) = 0$ because s_e is sublinear.

Consider now $u \in T_B(A, \bar{x}) \cap \text{cone}(L) \cap \nabla f(\bar{x})^{-1}(-\text{cone}M)$. There exist $(t_n) \downarrow 0$ and $u_n \rightarrow u$ such that

$$\bar{x} + t_n u_n \in A \cap (\bar{x} + \text{cone}L) \cap f^{-1}(f(\bar{x}) - \text{cone}M),$$

due to (4.3). Moreover, since $\bar{x} + t_n u_n \rightarrow \bar{x}$, we also have that $\bar{x} + t_n u_n \in U$ for n sufficiently large. Hence,

$$g(\bar{x} + t_n u_n) \geq g(\bar{x}) \Leftrightarrow s_e(f(\bar{x} + t_n u_n) - f(\bar{x})) \geq 0,$$

for all n sufficiently large. There exists a function $\alpha : X \rightarrow Y$ such that $\lim_{h \rightarrow 0} \alpha(\bar{x} + h) = \alpha(\bar{x}) = 0$, so that we can write

$$s_e(t_n \nabla f(\bar{x})(u_n) + t_n \alpha(\bar{x} + t_n u_n) \|u_n\|) \geq 0.$$

Dividing by t_n and passing to the limit, we get $s_e(\nabla f(\bar{x})(u)) \geq 0$, that is $\nabla f(\bar{x})(u) \notin -\text{int}K$. The proof is complete. \square

5 Concluding remarks

The type of metric inequalities on sets that we discuss and apply in this work is helpful in at least two aspects related to optimization problems. First of all, it is an appropriate replacement for some constraint qualification conditions when a new constraint is added. Secondly, it is naturally involved in penalization results when several constraints are considered. By their nature, as underlined, these metric inequalities are rather weak and this offers the perspective of applying the general principle

behind the results in this work to other research directions. Therefore, the metric relations we study here can be extended as well for deriving calculus rules for tangent cones by changing the conditions based on subregularities of mappings initiated in [4]. Moreover, penalization procedures for vectorial problems with multiple constraints based on results such as those in [18] and [6] are to be envisaged in future works.

Acknowledgement: The research of D.-E. Maxim and R. Strugariu was supported by the grant PN-III-P1-1.1-TE-2016-0868 of Romanian Ministry of Education and Research, CNCS–UEFISCDI.

References

- [1] H.H. Bauschke, J.M. Borwein, P. Tseng, *Bounded linear regularity, strong CHIP, and CHIP are distinct properties*, Journal of Convex Analysis, 7 (2000), 39–412.
- [2] T. Chelmuş, M. Durea, E.-A. Florea, *Directional Pareto efficiency: concepts and optimality conditions*, Journal of Optimization Theory and Applications, 182 (2019), 336–365.
- [3] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.
- [4] M. Durea, R. Strugariu, *Calculus of tangent sets and derivatives of set-valued maps under metric subregularity conditions*, Journal of Global Optimization, 56 (2013), 587–603.
- [5] M. Durea, R. Strugariu, *On some Fermat rules for set-valued optimization problems*, Optimization, 60 (2011), 575–591.
- [6] M. Durea, R. Strugariu, *Vectorial penalization for generalized functional constrained problems*, Journal of Global Optimization, 68 (2017), 899–923.
- [7] E.-A. Florea, D. Maxim, *Directional openness for epigraphical mappings and optimality conditions for directional efficiency*, Optimization, DOI: 10.1080/02331934.2019.1711085.
- [8] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, Berlin, 2003.
- [9] J.-B. Hiriart-Urruty, *From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality*, In: F.H. Clarke, V.F. Dem’yanov, F. Giannessi (eds) *Nonsmooth optimization and related topics*, Ettore Majorana International Science Series, vol. 43, Springer, pp. 219–240.
- [10] S.J. Li, K.W. Meng, J.-P. Penot, *Calculus rules for derivatives of multimaps*, Set-Valued and Variational Analysis, 17 (2009), 21–39.
- [11] S. Li, J.-P. Penot, X. Xue, *Codifferential calculus*, Set-Valued and Variational Analysis, 19 (2011), 505–536.
- [12] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory, Vol. II: Applications*, Springer, Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), Vol. 330 and 331, Berlin, 2006.

- [13] B.S. Mordukhovich, N. M. Nam, N. D. Yen, *Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming*, Optimization, 55 (2006), 685–708.
- [14] H.V. Ngai, M. Théra, *Metric inequality, subdifferential calculus and applications*, Set-Valued and Variational Analysis, 9 (2001), 187–216.
- [15] J.-P. Penot, *Metric estimates for the calculus of multimapping*, Set-Valued and Variational Analysis, 5 (1997), 291–308.
- [16] J.-P. Penot, *On the minimization of difference functions*, Journal of Global Optimization, 12 (1998), 373–382.
- [17] D. Tiba, C. Zălinescu, *On the necessity of some constraint qualification conditions in convex programming*, Journal of Convex Analysis, 11 (2004), 95–110.
- [18] J.J. Ye, *The exact penalty principle*, Nonlinear Analysis: Theory Methods and Applications, 75 (2012), 1642–1654.