Controllability, Stabilization and Inverse Problems for Parabolic Systems

PhD Thesis - Abstract
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Introduction

The aim of this thesis is to study the controllability, stabilization and inverse stability estimates associated to initial boundary value problems for parabolic equations and systems. The control problems we address, as well as the inverse estimates, involve control and observation operators distributed in subdomains. When such operators are considered, a key tool in the approach is represented by the global Carleman estimates which will play a central role in our work.

0.1. A general description of the field

Controllability of heat equation with boundary or internal controls was established by D. Russell [90] in connection to similar problem for the wave equation and later by G. Lebeau and L. Robbiano [66] without geometric condition. Controllability for general linear parabolic equations, with internally distributed controls supported in a subdomain, was established by O. Yu. Imanuvilov using appropriate global Carleman estimates for the adjoint equation (see [57] for an introduction to the field). Local controllability of nonlinear equations or systems may be deduced from controllability of the linearized system. Such an approach, passing through the linearized problem, usually gives small time local controllability results for the initial nonlinear problem. This step is typically performed with the aid of a fixed point argument based on some corresponding fixed point theorem like Kakutani Theorem for multivalued functions.

Carleman estimates are fundamental not only for the study of the linear problem in the linearizations procedure, but also for the step of verifying certain hypotheses in appropriate fixed point theorems (see for example [11] or the survey [15]). Growth conditions at infinity for the nonlinearity need to be imposed for obtaining global controllability results (see [54, 49]).

An argument based on the regularizing properties of the parabolic flow gives Carleman estimates in the $L^\infty - L^1$ framework and this is useful for deriving more regularity for the controls in controllability problems (see [16]). This kind of argument may also be found in [39] where L^∞ estimates for the control are established; this result is then used for local controllability in a L^∞ neighborhood of a particular trajectory to a reaction-diffusion system. Such estimates are involved in situations when state constraints are imposed, like positivity in the models of reaction-diffusion processes [39, 40, 65].

Controllability of semilinear systems by a reduced number of controls is a challenging problem and positive results may be obtained under appropriate conditions on the coupling terms. The study of controlled systems of parabolic equations with fewer controls than equations need appropriate Carleman estimates, with partial observations for the

adjoint to the linearized system. This is for example the case of systems with cascade-like couplings, with one control, studied in [**59**].

In some cases, like constant or time dependent coupling coefficients, one may provide algebraic conditions of Kalman type involving the coupling operator and the control operator, in order to obtain appropriate observability inequalities. In this direction we refer to [**4**, **6**].

When a good coupling is not verified, in the sense that the linearized system is not controllable, an issue to exploit the nonlinearity is the return method of J.-M. Coron, which consists in linearization along special solutions, constructed in such a way to fulfill coupling requirements for the linearized system (see [**38**, **39**, **40**]).

A recent paper considering coupled linear systems (not only parabolic) and corresponding observation estimates is the paper of E. Zuazua and P. Lissy [**76**] where the equations in the system are linearly coupled with constant coefficients in the dominant part and/or in the zero order terms; algebraic Kalman conditions for observability are established.

The cost of approximate controllability relies also on a refined analysis of the Carleman estimates and the dependence of the constants appearing in the estimates on the time interval of observation (see [**55**]). Consequences of such refined estimates are unique continuation results at initial time (see [**70**, **8**]).

Stabilization by feedback controls for parabolic systems may pass also through a linearizations procedure. The stabilization of the linearized system needs a Riccati type approach for obtaining robust feedback laws, which are appropriate, at least locally, for the nonlinear model. We mention for example the papers of V. Barbu and G. Wang [**12**, **28**] concerning stabilization of parabolic equations and the papers of V. Barbu, I. Lasiecka, R. Triggiani [**27**, **21**, **22**, **23**] for the stabilization of Navier-Stokes equations (with either internal or boundary controls). In these papers the authors use Kalman type conditions for finite dimensional projections of the equations on unstable subspaces in order to construct feedback laws for the linearized problem, as a basis for solving an appropriate Riccati equation. These Kalman conditions are obtained through some unique continuation property for systems of eigenfunctions of the elliptic part.

An alternative approach to the Riccati equations is based on Lyapunov equation. A Lyapunov equation associated to a linear problem gives appropriate equivalent norms in corresponding function spaces, in which one is able to show local stabilization of the nonlinear problem (see [**67**, **68**]). In this approach one needs unique continuation properties for systems of parabolic-elliptic equations which may be also derived from Carleman estimates.

Estimates for the cost of approximate controllability are useful also in problems of feedback stabilization of nonstationary solutions to parabolic systems, as it is done in [**26**, **69**].

For a recent book dedicated to parabolic and parabolic like problems (fluid dynamics models, Navier-Stokes equations) we refer to the monograph of V.Barbu [**18**].

Inverse problems appear from practice and thus are of great importance from the point of view of applications. Inverse problems refer to models in which some quantities are

not known, like coefficients or sources, and one would like to obtain informations about these quantities by some extra measurements on the solution. Stability estimates for the unknown quantities are useful in developing stable algorithms for the approximation and numerical approach. We mention here the book of M. Choulli [36] for an introduction to the field. We will address in this thesis only stability estimates in L^p norm for the source in parabolic systems and the starting point is the paper of O. Yu. Imanuvilov and M. Yamamoto [61].

Regularity of the solutions to

$$(0.1) \qquad\qquad y' + Ay = f, y(0) = 0, \, t \in (0, T)$$

with A the L^p realization of parabolic operator L with homogeneous boundary conditions and sources from various function spaces based on the classical Hölder and Sobolev spaces, is important in the analysis of the control and inverse problems. The classical reference for existence and regularity of solutions to parabolic problems with $f \in L^p(Q) \simeq L^p(0, T; L^p(\Omega))$ is the monograph by O. A. Ladyzenskja, V. A. Solonnikov, N. N. Uralceva [64], where maximal regularity is obtained in the anisotropic Sobolev spaces $W_p^{2,1}(Q)$. The regularity of solutions to abstract parabolic problems and estimates in real interpolation spaces, were considered in the paper of Gabriella Di Blasio [46]; there, for $f \in L^q(X), q \in (1, \infty)$ one obtains $S * f \in W^{\theta,q}(X), \theta \in (0,1)$ and $S * f \in L^q(D_A(\theta, q)), \theta \in (0,1)$ (here $D_A(\theta, q) = (X, D(A))_{\theta,q}$ and $W^{\theta,q}(X)$ is a vector valued Sobolev-Slobodeckii space).

The existence and maximal regularity in concrete parabolic problems with $X = L^p(\Omega)$ is established by W. von Wahl in [100], where estimates for $S*f \in L^q(D(A)), D_t(S*f) \in L^q(X)$ in terms of norm of $f \in L^q(0, T; L^p(\Omega))$ with $q, p > 1$ are obtained by applying a refined study of A. Benedek, A.P.Calderón, R. Panzone [29] on the convolution of operators, using ideas from the theory of singular integrals.

When dealing with parabolic problems with nonhomogeneous boundary conditions, a study of maximal regularity in $L^q(L^p)$ spaces was established by P. Weidemaier [102, 103].

Maximal regularity in $L^q(X)$ for an abstract parabolic problem is deeply related to the geometry of X and properties of operator A. More precisely, if X is *UMD* space (a space having the property that the vector valued Hilbert transform is bounded in $L^q(X)$) and A is sectorial with bounded imaginary powers, $A \in BIP(X, \theta)$, with spectral angle $\theta < \frac{\pi}{2}$, then equation (0.1) has maximal regularity property: $y \in W^{1,q}(X) \cap L^q(D(A))$. We refer here to the monograph of C. Martinez Carracedo, M. Sanz Alix [79], Chapter 8 and the references therein. An essential ingredient in the approach of such problems is a theorem of G. Dore and A. Venni characterizing invertibility of sums of operators in *BIP* class.

The *BIP* class is important in our presentation of parabolic regularity; it allows to characterize the domains of powers of positive operators defined by elliptic operators with boundary conditions, as complex interpolation spaces. In such situation these are closed subspaces of Bessel potential spaces (this important result is due to R. T. Seeley [94]). Then, by using an argument based on extension operators one may relate these

spaces to Sobolev-Slobodeckii spaces. Another ingredient we use in studying regularity is represented by convolution estimates in $L^r(D(A^\gamma))$, by using estimates which are specific to analytic semigroups, in domains of fractional powers of the generating operator.

As we are interested in L^p realizations of elliptic operators in bounded domains, we mention that the boundedness of imaginary powers of such operators was proved by R. T. Seeley in [96] by using a representation of the resolvent and the theory of pseudodifferential operators ([95], [93]). A more direct approach to such results was given by J. Prüss and H. Sohr in [88] (see also Th. 12.1.12 in [79]). We also mention here the paper by R. Denk, G. Dore, M. Hieber, J. Prüss, A. Venni [41] for a study of elliptic operators with Hölder coefficients in principal part, in connection to the $\mathcal{H}^\infty$ calculus and the *BIP* property.

Concernig classical Gagliardo-Nirenberg inequalities for Sobolev-Slobodeckii spaces, in the most general framework, we refer to the papers of H. Brezis and P. Mironescu [32], [33].

0.2. Main results

Part 1. Stabilization of coupled parabolic systems

Chapter 1: Feedback stabilization with one simultaneous control for systems of parabolic equations. We study the local feedback stabilization of systems of parabolic equations in a bounded domain $\Omega \subset \mathbb{R}^N, N \in \{2,3\}$ with C^2 boundary, under only one internally distributed control, supported in a bounded subdomain $\omega \subset\subset \Omega$ and acting simultaneously in both equations. We established a result of feedback stabilization based on approximate controllability for the linearized system. This implies exact controllability for the finite dimensional system and, consequently, this has the property of complete stabilization. We may thus construct a feedback law stabilizing the finite dimensional part and then prove that this is stabilizing the full linearized system. The fact that the feedback law constructed in the linear case is also stabilizing the nonlinear system is proved by using the solution of an appropriate Lyapunov equation. The system we study is the following,

$$(0.2) \quad \begin{cases} y_t - d_1 \Delta y = f(y,z) + f_1 + \psi_\omega u, & \text{in } (0,T) \times \Omega, \\ z_t - d_2 \Delta z = g(y,z) + g_1 + \psi_\omega u, & \text{in } (0,T) \times \Omega, \\ y(t,x) = 0, \quad z(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad z(0,x) = z_0(x), & \text{in } \Omega. \end{cases}$$

where $d_1, d_2 \in \mathbb{R}_+$ are the diffusion coefficients, $f, g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are C^∞ coupling nonlinearities, $f_1, g_1 \in L^\infty(\Omega)$, $\psi_\omega \in C^\infty(\Omega)$, supp $\psi_\omega = \overline{\omega}$, $\psi_\omega > 0$ in ω. $u(t, \cdot)$ is the control which belongs to $L^2(\omega)$ and by $\psi_\omega u$ we denote the extension by 0 of u to Ω multiplied by ψ_ω. In the following, $Y, \overline{Y}, Y_0$ denote vector functions $(y,z)^\top, (\overline{y}, \overline{z})^\top, (y_0, z_0)^\top$.

Let $(\overline{y}, \overline{z}) \in (L^\infty(\Omega))^2$ be a stationary state of the system. Then the controlled linearized system is

(0.3)
$$\begin{cases} \xi_t - d_1\Delta\xi = a(x)\xi + b(x)\eta + \psi_\omega u, & \text{in } (0,T) \times \Omega, \\ \eta_t - d_2\Delta\eta = c(x)\xi + d(x)\eta + \psi_\omega u, & \text{in } (0,T) \times \Omega, \\ \xi = 0, \quad \eta = 0, \quad \text{on } (0,T) \times \partial\Omega, \\ \xi(0) = \xi_0, \quad \eta(0) = \eta_0, \quad \text{in } \Omega, \end{cases}$$

where

$$a(x) := \frac{\partial f}{\partial y}(\overline{y}, \overline{z}), \quad b(x) := \frac{\partial f}{\partial z}(\overline{y}, \overline{z}), \quad c(x) := \frac{\partial g}{\partial y}(\overline{y}, \overline{z}), \quad d(x) := \frac{\partial g}{\partial z}(\overline{y}, \overline{z}).$$

Let H be the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ and consider the operators

$$A : D(A) \subset H \longrightarrow H, \quad D(A) = (H_0^1(\Omega) \cap H^2(\Omega))^2, \quad A = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix},$$

$$A_0 : D(A_0) = H \longrightarrow H, \quad A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B : L^2(\omega) \longrightarrow H, \quad Bu = \begin{pmatrix} \psi_\omega u \\ \psi_\omega u \end{pmatrix} \text{ and}$$

$$\mathcal{A} := A + A_0.$$

Denoting by

$$\gamma(x) = [a(x) + b(x) - c(x) - d(x)],$$

$$\alpha(x) := \left[(c(x) - a(x)) - \frac{d_1\gamma(x)}{d_2 - d_1} \right] = \left[(b(x) - d(x)) - \frac{d_2\gamma(x)}{d_2 - d_1} \right]$$

and

$$L_T v := \Delta v + \frac{\gamma(x)}{d_1 - d_2} v, \ D(L_T) = H_0^1(\Omega) \cap H^2(\Omega),$$

the stabilization result that will be proved for the linearized system is the following:

THEOREM 1.1. *Suppose that the diffusion coefficients are distinct $d_1 \neq d_2$ and one of the following assumptions is true:*
  - *α is not identically constant in ω, or*
  - *α is a constant in Ω and $0 \notin \sigma(L_T)$.*

Then the following conclusions hold:
  (i) *The operator $\mathcal{A} = A + A_0$ has compact resolvent and generates an analytic semigroup in H;*
  (ii) *The linear system (0.3) is approximately controllable in any time T;*
  (iii) *For any $\delta > 0$ there exist $C = C(\delta) > 0$, a finite dimensional subspace $U \subset L^2(\omega)$ and a linear continuous operator $K \in L(H,U)$ such that the operator $\mathcal{A} + BK$ generates an analytic semigroup of negative type satisfying*

(0.4)
$$\|e^{t(\mathcal{A}+BK)}\|_H \leq Ce^{-\delta t}, \quad t > 0.$$

The main result of the chapter concerning the stability around the stationary state of the nonlinear system is

THEOREM 1.2. *In the hypotheses of the above Theorem, we have that there exist $\varepsilon > 0$, $\delta > 0$, $C > 0$, $\tau > 0$ such that if $\|y_0 - \overline{y}\|_{L^\infty \cap H^1(\Omega)} + \|z_0 - \overline{z}\|_{L^\infty \cap H^1(\Omega)} \leq \varepsilon$ then,*

taking in (0.2) *the feedback constructed in the above Theorem*

$$(0.5) \qquad\qquad u = K(Y - \overline{Y}),$$

we have local exponential stabilization:

$$(0.6) \qquad \begin{aligned} &\|Y(t) - \overline{Y}\|_{H^s(\Omega)} \leq Ce^{-\delta t}\|Y_0 - \overline{Y}\|_{H^1 \cap L^\infty(\Omega)}, \, t > \tau, s \in [0, 2], \\ &\|Y(t) - \overline{Y}\|_{H^1 \cap L^\infty(\Omega)} \leq Ce^{-\delta t}\|Y_0 - \overline{Y}\|_{H^1 \cap L^\infty(\Omega)}, \, t > 0. \end{aligned}$$

Chapter 2: Internal feedback stabilization for parabolic systems coupled in zero and first order terms. We study the local feedback stabilization for systems of parabolic equations in one dimension, *i.e.* on a bounded interval $\Omega \subset \mathbb{R}$. The equations are coupled in either first or zero order terms and we consider general boundary conditions, arbitrarily mixing Dirichlet, Neumann and Robin conditions. Under algebraic conditions of Kalman type concerning the coupling matrices of coefficients, we establish finite dimensional feedback stabilization with internal controls distributed in a subdomain $\omega \subset \Omega$ and acting in part of the equations through a control matrix.

The strategy is to first linearize the nonlinear system around the stationary state and to prove approximate controllability for it. For systems of two equations we treat the case of couplings in both zero and first order terms and homogeneous Dirichlet boundary conditions. For systems of $n \geq 3$ equations, we will treat separately the cases of first or zero order couplings. The approximate controllability is obtained by proving the unique continuation property for the adjoint system under corresponding Kalman type conditions satisfied by the coupling matrix and the control matrix.

We consider an abstract formulation for the given problem as an evolution problem in a Hilbert space. With the result of approximate controllability for the linearized system at hand, we use a spectral decomposition of this Hilbert space with respect to the elliptic operator in a direct sum of closed and invariant subspaces for the semigroup. Moreover, one of these subspaces is finite dimensional, corresponding to the eigenvalues with positive real part (that is the unstable subspace) and the other one is infinite dimensional but stable. With this decomposition of the space we consider the controlled system projected onto these subspaces and we study the controllability of the finite dimensional system. The approximate controllability gives the exact controllability for the system in any time in the finite dimensional subspace and, consequently, complete stabilization for it. We stabilize by a feedback control the finite dimensional projection of the system and we prove, using the norm given by the solution to an appropriate Lyapunov equation, that this finite dimensional feedback control stabilizes the whole nonlinear system.

For $\omega \subset\subset \Omega$ an open nonempty subset of Ω, we consider the controlled parabolic system

$$(0.7) \qquad \begin{cases} D_t y - Ly + F(D_x y, y) = g + B\chi_\omega u, & t > 0, x \in \Omega, \\ (\mathbf{BC}) : \begin{cases} \Gamma_1 D_x y(t, 0) + \Gamma_2 y(t, 0) = 0, \\ \Gamma_3 D_x y(t, l) + \Gamma_4 y(t, l) = 0, \end{cases} & t > 0, \\ y(0, x) = y^0(x), & x \in \Omega, \end{cases}$$

where $y = (y_1, \cdots, y_n)^\top$, $Ly = (Ly_1, \cdots, Ly_n)^\top$ with L an uniformly elliptic operator of second order,

$$Ly = D_x^2 y + \eta_1(x)D_x y + \eta_0(x)y, \quad \eta_0 \in L^\infty(\Omega), \eta_1 \in W^{1,\infty}(\Omega).$$

The equations are coupled in either first or zero order terms through a C^∞ function, $F(\zeta, y) = (f_1(\zeta, y), \cdots, f_n(\zeta, y))^\top$, $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and in the right-hand side the free term $g = (g_1, \cdots, g_n)^\top$ is $L^\infty(\Omega)$. The control is given by $B\chi_\omega u$, $B \in \mathcal{M}_{n \times m}(\mathbb{R})$, $u \in L^2((0,T); [L^2(\omega)]^m)$, where $\chi_\omega u$ is the extension of u by 0 to the whole Ω. For a general formulation of these boundary conditions we choose diagonal matrices $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in \mathcal{M}_{n \times n}(\mathbb{R})$ with the properties

$$(0.8) \qquad \begin{aligned} &\Gamma_i = \mathrm{diag}(\gamma_i^j)_{j=\overline{1,n}}, \forall i = \overline{1,4}, j = \overline{1,n}, \\ &\mathrm{rank}[\Gamma_1, \Gamma_2] = \mathrm{rank}[\Gamma_3, \Gamma_4] = n. \end{aligned}$$

We consider a stationary solution of the uncontrolled system, denoted by $\overline{y}$. The linear system obtained through linearization of the nonlinear system around the stationary state is

$$(0.9) \qquad \begin{cases} D_t w - Lw + A_1(x)D_x w + A_0(x)w = B\chi_\omega u, & t > 0, x \in \Omega, \\ \Gamma_1 D_x w(t,0) + \Gamma_2 w(t,0) = \Gamma_3 D_x w(t,l) + \Gamma_4 w(t,l) = 0 & t > 0, \end{cases}$$

where $A_0(x), A_1(x) \in \mathcal{M}_{n \times n}(\mathbb{R})$,

$$A_1(x) = \left(\frac{\partial f_i}{\partial \zeta_j}(D_x \overline{y}, \overline{y}) \right)_{i,j=\overline{1,n}}, A_0(x) = \left(\frac{\partial f_i}{\partial y_j}(D_x \overline{y}, \overline{y}) \right)_{i,j=\overline{1,n}}.$$

The aim of the chapter is to find a control in feedback form $u = K(y - \overline{y})$, such that it stabilizes the controlled system (0.7) around the stationary state $\overline{y}$ with respect to a topology to be precised later.

We construct the abstract formulation for the given problem as an evolution problem in a Hilbert space:

$H = [L^2(\Omega)]^n$ is the Hilbert space and

$\mathbb{A} : D(\mathbb{A}) \subset H \longrightarrow H$, $D(\mathbb{A}) = \{y \in [H^2(\Omega)]^n | y$ satisfies $(\mathbf{BC})$ of $(0.7)\}$, $\mathbb{A}y = Ly$,

$$\mathcal{A} : D(\mathcal{A}) = D(\mathbb{A}), \mathcal{A}y = Ly - A_1(x)D_x y - A_0(x)y,$$

with the control operator

$$\mathbb{B} : L^2(\omega)^m \longrightarrow H, \quad \mathbb{B}u = B\chi_\omega u.$$

For a system of two coupled equations for $w = (w_1, w_2)$, with differential operator L with constant coefficients $\eta_0, \eta_1 \in \mathbb{R}$ and possibly nonconstant couplings, under Dirichlet homogeneous boundary conditions:

$$(0.10) \qquad \begin{cases} D_t w_1(t,x) - Lw_1(t,x) & +a(x)D_x w_1 + b(x)D_x w_2 \\ & +\alpha(x)w_1 + \beta(x)w_2 = \chi_\omega u, & t > 0, x \in \Omega, \\ D_t w_2(t,x) - Lw_2(t,x) & +c(x)D_x w_1 + d(x)D_x w_2 \\ & +\gamma(x)w_1 + \delta(x)w_2 = 0, & t > 0, x \in \Omega, \\ w(t,0) = w(t,l) = 0, & t > 0. \end{cases}$$

If we consider the notations,

$$(0.11) \qquad h(x) := \frac{\gamma(x) - c'(x)}{c(x)},$$

$$k(x) := -h^2(x) - h'(x) + [\eta_1 - d(x)]h(x) - \eta_0 + \delta(x) - d'(x),$$

for $c(x) \neq 0$ in ω, we have the following results concerning the approximate controllability for the above linear systems:

THEOREM 2.1. *For the linear system* (0.10), *with* $\eta_0, \eta_1 \in \mathbb{R}$, *if* $c(x) \neq 0$ *for* $x \in \omega$ *and one of the following hypotheses are verified*

(H1) the coefficients of the system are constants in the whole domain,

$$a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R};$$

(H2) the coupling coefficients are continuous in Ω, *maybe nonconstant, and the function* $k = k(x)$ *is not constant in* ω;

then the linear system (0.10) *is approximately controllable in time* T.

For the system (0.9) we have the following results concerning the approximate controllability:

THEOREM 2.2. *Consider the linear system* (0.9) *with constant coefficients* η_0, η_1 *and with constant couplings of order zero,* $A_1 \equiv 0$. *If the following Kalman condition holds,*

$$(0.12) \qquad \text{rank } [A_0|B] = n,$$

then the linear system (0.9) *is approximately controllable in time* T.

Regarding the case of constant couplings of order one, we have the following result:

THEOREM 2.3. *Consider the linear system* (0.9) *with constant coefficients* η_0, η_1 *and with constant couplings of order one,* $A_0 \equiv 0, A_1 \in \mathcal{M}_{n \times n}(\mathbb{R})$. *Suppose also that the following algebraic conditions concerning coupling matrix and matrices entering the boundary conditions are satisfied:*

$$(0.13) \qquad \text{rank } [A_1|B] = n,$$

$$(0.14) \qquad \ker B^\top \cap \ker(\Gamma_2 + \Gamma_1(A_1^\top + \eta_1 I)) \cap \ker(\Gamma_4 + \Gamma_3(A_1^\top + \eta_1 I)) = \{0\}$$

then the linear system (0.9) *is approximately controllable in time* T.

In either of the cases when approximate controllability is verified, we prove the following feedback stabilization result for the linearized system:

THEOREM 2.4. *For the linear system* (0.9), *in the framework of either of Theorems 2.1, 2.2 or 2.3, for any* $\delta > 0$ *there exist* $C = C(\delta) > 0$, *a finite dimensional subspace* $U \subset L^2(\omega)$ *and a bounded linear operator* $K \in L(H, U)$ *such that the operator* $\mathcal{A} + \mathbb{B}K$ *generates an analytic semigroup of negative type that satisfies*

$$(0.15) \qquad \|e^{(\mathcal{A} + \mathbb{B}K)t}\|_H \leq Ce^{-\delta t}, \quad t > 0.$$

Based on the stabilization results in the linear case, we prove by using an argument related to Lyapunov equation, a local feedback stabilization result:

THEOREM 2.5. *Consider the nonlinear system* (0.7) *and suppose that we are in the framework of either of Theorems 2.1, 2.2 or 2.3 for the linearized system.*

Let $\nu \in (\frac{3}{4}, 1)$. Then there exist $\varepsilon > 0$, $\delta > 0$, $C > 0$, such that if $y^0 \in H^{2\nu}(\Omega)$ verifies the boundary conditions (**BC**) *in* (0.7) *and $\|y^0 - \overline{y}\|_{H^{2\nu}(\Omega)} \leq \varepsilon$ then, taking in* (0.7) *the feedback constructed in Theorem 2.4*

$$(0.16) \qquad\qquad u = K(y - \overline{y})$$

one has exponential stabilization:

$$(0.17) \qquad \|y(t) - \overline{y}\|_{H^{2\nu}(\Omega)} + \|y(t) - \overline{y}\|_{L^\infty(\Omega)} \leq C e^{-\delta t} \|y^0 - \overline{y}\|_{H^1(\Omega)}, \, t > 0.$$

Part 2. Controllability of coupled parabolic systems

Chapter 3: Internal controllability of parabolic systems with star and tree like couplings. In this chapter we consider semilinear systems of parabolic equations coupled in zero order terms. We are interested in controllability of such systems to stationary solutions by only one control distributed in a subdomain and acting in only one of the equations. The key hypotheses insuring local controllability refer to the structure of the couplings, which describe either a star or a tree type graph, and to the support of the coupling functions or, in the linear case, to the support of the coupling coefficients.

The strategy for proving the controllability result relies on the linearization of the nonlinear system around a stationary state. The key step is obtaining the null controllability for this linear system by using an observability inequality for the adjoint system. This observability inequality is consequence of an appropriate global Carleman estimate. This in turn is obtained by combining Carleman estimates for each of the equation, but relying on diferent auxiliary functions, which are in a particular order relation, made possible by the special structure of the system. The idea of using different auxiliary functions in Carleman estimates is inspired by the work of G. Olive [85] concerning controllability of parabolic systems with controls acting in different subdomains.

Passing from the linearized system to the nonlinear one needs an L^∞ framework for the controlability of the linear system because the Carleman estimates we obtain are sensitive to zero order perturbations of the system. More regularity of the controls in the linearized problem is obtained as in the work of V. Barbu [16] (see also [39]) by using regularizing properties of the parabolic flow in a bootstrap argument. This allows an approach to the controllability of the nonlinear system by a fixed point argument, based on Kakutani theorem, as in the work of J.-M. Coron, S. Guerrero and L. Rosier [39] or [11]. In fact the proof of this step follows the same lines as in [39] where the return method is used and the linearization is performed around a particular trajectory, such

that the linearized system is well coupled; this also is a situation where an L^∞ framework for the controllability is necessary by the same reason as in the case we are considering.

In the first part of the chapter we study systems of parabolic equations with star-like couplings which refer to the situation where y_k is actuated in the corresponding parabolic equation through a nonlinearity depending only on y^0, y^k. Such a star-like coupled system has the form:

$$(0.18) \quad \begin{cases} D_t y_0 - \Delta y_0 = g_0(x) + f_0(x, y_0) + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\ D_t y_i - \Delta y_i = g_i(x) + f_i(x, y_0, y_i), \ i \in \overline{1, n}, & \text{in } (0, T) \times \Omega, \\ y_0 = ... = y_n = 0, & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, & \text{in } \Omega, \end{cases}$$

where $g_i \in L^\infty(\Omega)$, $i = \overline{0, n}$. Concerning the coupling terms we assume the following:

(H1) $f_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are C^1 functions and there exist $\omega_1, ... \omega_n \subset \Omega$ open nonempty subsets of Ω such that

$$(0.19) \quad (\omega_i \cap \omega_0) \setminus \bigcup_{j \neq 0, i} \omega_j \neq \emptyset, \ \forall i \in \overline{1, n},$$

and for all $i \in \overline{1, n}$ we have

$$(0.20) \quad f_i(x, y_0, y_i) = 0 \ \forall x \in \Omega \setminus \omega_i, \ y_0, y_i \in \mathbb{R};$$

(H2) The following coupling condition holds:

$$(0.21) \quad \text{supp} \frac{\partial f_i}{\partial y_0}(x, \overline{y}_0(x), \overline{y}_i(x)) \cap \left\{ (\omega_i \cap \omega_0) \setminus \overline{\bigcup_{j \neq 0, i} \omega_j} \right\} \neq \emptyset,$$

The control function is $u : [0, T] \times \omega_0 \longrightarrow \mathbb{R}$, acting in the equation of y_0 and controlling the other components of the solution, $y_1, ..., y_n$, through the action of y_0 in each equation, on the corresponding subdomain $\omega_i, i \in \overline{1, n}$.

We consider first a controlled linear system which will appear through a linearization procedure around a stationary state $\overline{y} = (\overline{y}_0, ..., \overline{y}_n) \in [L^\infty(\Omega)]^{n+1}$:

$$(0.22) \quad \begin{cases} D_t z_0 - \Delta z_0 = c_0(t, x) z_0 + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\ D_t z_i - \Delta z_i = a_{i0}(t, x) z_0 + c_i(t, x) z_i, \ i \in \overline{1, n}, & \text{in } (0, T) \times \Omega, \\ z_0 = ... = z_n = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

For $M, \delta > 0$, and open subsets $\underline{\omega_i} \subset\subset (\omega_i \cap \omega_0) \setminus \bigcup_{j \neq 0, i} \omega_j$ we introduce the following classes of coefficients sets:

$$(0.23) \quad \mathcal{E}_{M, \delta, \{\underline{\omega_i}\}_i} = \left\{ E = \{a_{i0}, c_j\}_{i \in \overline{1, n}, j \in \overline{0, n}} : a_{i0}, c_j \in L^\infty(Q), \|a_{i0}\|_{L^\infty}, \|c_j\|_{L^\infty} \leq M; \right.$$
$$\left. a_{i0} = 0 \text{ in } Q \setminus Q_{\omega_i}, \text{ and } |a_{i0}| \geq \delta \text{ on } Q_{\underline{\omega_i}}, \forall i, j \right\}.$$

We prove first that such linear systems with coefficients in $\mathcal{E}_{M, \delta, \{\underline{\omega_i}\}_i}$ are null controllable with norm L^2 and L^∞ of the control uniformly bounded by a constant $C = C(M, \delta, \{\underline{\omega_i}\}_i)$.

In order to achieve this goal we consider the adjoint system:

$$(0.24) \quad \begin{cases} -D_t p_0 - \Delta p_0 = c_0(t,x) p_0 + \sum_{i=1}^n a_{i0}(t,x) p_i, & (0,T) \times \Omega, \\ -D_t p_i - \Delta p_i = c_i(t,x) p_i, \ i \in \overline{1,n}, & (0,T) \times \Omega, \\ p_0 = \dots = p_n = 0, & (0,T) \times \partial\Omega, \end{cases}$$

and we prove an observability inequality as consequence of an appropriate Carleman estimate. We consider the open subsets

$$\tilde{\omega}_j \subset\subset \underline{\omega}_j$$

and denote as above by $Q_{\tilde{\omega}_j} = (0,T) \times \tilde{\omega}_j$.

The Carleman estimates we establish need a particular choice of auxiliary and weight functions. We consider

$$(0.25) \qquad \overline{\alpha}(t) = \overline{\alpha}^\lambda(t) := \frac{e^{\lambda\overline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)}, \quad \underline{\alpha}(t) = \underline{\alpha}^\lambda(t) := \frac{e^{\lambda\underline{\psi}} - e^{1.5\lambda\overline{\psi}}}{t(T-t)},$$

$$(0.26) \qquad \overline{\psi} = \sup_{x\in\Omega}\sup_{i=\overline{0,n}} \psi_i(x) + \epsilon, \qquad \underline{\psi} = \inf_{x\in\Omega}\inf_{i=\overline{0,n}} \psi_i(x) - \epsilon,$$

with the family of auxiliary functions $\{\psi_i\}$ similar to the corresponding function in the classical Carleman estimates, but each of them concentrating its critical points in $\tilde{\omega}_i$ and $0 < \epsilon < \inf \psi_i, i \in \overline{0,n}$. A supplementary technical assumption is

$$(0.27) \qquad \frac{\sup \psi_i}{\inf \psi_i} < \frac{8}{7}, \forall i = \overline{0,n}.$$

The fundamental result in Chapter **??** concerns the Carleman estimate for our problem:

THEOREM 3.1. *There exist constants λ_0, s_0 such that for $\lambda > \lambda_0$ there exists a constant $C > 0$ depending on $(M, \delta, \{\underline{\omega}_i\}_i, \lambda)$, such that, for any $s \geq s_0$, the following inequality holds:*

$$(0.28) \qquad \begin{aligned} &\int_Q (|D_t p|^2 + |D^2 p|^2 + |Dp|^2 + |p|^2) e^{2s\underline{\alpha}} dx dt \\ &\leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2s\overline{\alpha}} dx dt \end{aligned}$$

for all $p \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ solution of (0.24).

Moreover, there exist $m_0 \in \mathbb{N}$ and $\delta_1 > 0$ such that for the homogeneous adjoint system (i.e. taking $g \equiv 0$), we have the following $L^\infty - L^2$ Carleman estimate

$$(0.29) \qquad \|p e^{(s+m_0\delta_1)\underline{\alpha}}\|_{L^\infty(Q)} \leq C\|p_0 e^{s\overline{\alpha}}\|_{L^2(Q_{\omega_0})}.$$

The main controllability result concerning for linear system (0.22) is the following

THEOREM 3.2. *Consider system (0.22) with coefficients in $\mathcal{E}_{M,\delta,\{\underline{\omega}_i\}_i}$. Then there exists a constant $C = C(M, \delta, \{\underline{\omega}_i\}_i)$ such that for all $z^0 \in H$ there exists $u^* \in L^2(0,T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ which drives in 0 the corresponding solution $z = z^{u^*}$ to (0.22) with $z(0, \cdot) = z^0$, i.e. $z(T, \cdot) = 0$, and satisfies the norm estimate*

$$(0.30) \qquad \|u^* e^{-s\overline{\alpha}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C\|z^0\|_{L^2(\Omega)}.$$

Regarding the local controllability for the nonlinear system with star-like couplings we have the following result:

THEOREM 3.3. *Suppose $\overline{y}$ is a stationary state to the uncontrolled system* (0.18) *and that the functions f_j, $j \in \overline{0,n}$ satisfy hypotheses* (H1), (H2). *Then, for all $\beta_0 > 0$ there exist $\zeta_0 = \zeta_0(\beta_0) > 0$ and $C = C(\beta_0, \{\underline{\omega}_i\}_i, \overline{y})$ such that if $\|y^0 - \overline{y}\|_{L^\infty(\Omega)} < \zeta_0$ there exists a control $u \in L^\infty(Q_{\omega_0})$ and the corresponding solution $y = y^u$ to* (0.18) *such that*

$$\|u\|_{L^\infty(Q_{\omega_0})} \le C\|y^0 - \overline{y}\|_{L^\infty(\Omega)}$$

with

$$y^u(T, \cdot) = \overline{y} \text{ and } \|y^u(t, \cdot) - \overline{y}\|_{L^\infty(\Omega)} \le \beta_0, \ t \in [0, T].$$

Now, we describe in the following what we mean by a linear system with tree-like couplings. This would be a parabolic system of the form

(0.31)
$$\begin{cases} D_t z_0 - \Delta z_0 = c_0(t, x)z_0 + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\ D_t z_i - \Delta z_i = a_{i\mathbf{k}(i)}(t, x)z_{\mathbf{k}(i)} + c_i(t, x)z_i, \ i \in \overline{1, n}, & \text{in } (0, T) \times \Omega, \\ z_0 = \dots = z_n = 0, & \text{on } (0, T) \times \partial\Omega, \\ z(0, \cdot) = z^0, & \text{in } \Omega, \end{cases}$$

with the following assumptions on the function $\mathbf{k} : \{1, \dots, n\} \to \{1, \dots, n\}$:

(0.32) $\forall i \in \{1, \dots, n\}, \exists m = m(i), 1 \le m \le n - 1, (\mathbf{k}\circ)^m(i) = \mathbf{k} \circ \dots \circ \mathbf{k}(i) = 0.$

Denote by

$$\mathbf{I}_j = \mathbf{k}^{-1}(j) = \{i \in \overline{1, n} : \mathbf{k}(i) = j\}.$$

Fix now a family of open subsets $\omega_i \subset \Omega, i \in \overline{1, n}$ such that

(0.33) $D_i := \omega_i \cap \omega_{\mathbf{k}(i)} \cap \dots \cap \omega_{(\mathbf{k}\circ)^{m(i)}} \ne \emptyset.$

(0.34) $D_i \setminus \bigcup_{j \ne i, \mathbf{k}(j) = \mathbf{k}(i)} \omega_j \ne \emptyset.$

Choose further a family of open subsets $\{\underline{\omega}_j\}_{j \in \overline{0,n}}$ with the properties

(0.35) $\underline{\omega}_0 \subset\subset \omega_0, \quad \underline{\omega}_i \subset\subset D_i \setminus \bigcup_{l \ne i, \mathbf{k}(l) = \mathbf{k}(i)} \omega_l$

(0.36) $\underline{\omega}_i \subset\subset \underline{\omega}_{\mathbf{k}(i)} \subset\subset \underline{\omega}_0, \ i \in \overline{1, n}.$

For $M, \delta > 0$, and the family of open subsets described above $\{\underline{\omega}_i\}_i$, we introduce the following classes of coefficients sets:
(0.37)

$$\mathcal{E}_{M,\delta,\{\underline{\omega}_i\}_i,\mathbf{k}} = \Big\{ E = \{a_{i\mathbf{k}(i)}, c_j\}_{i \in \overline{1,n}, j \in \overline{0,n}} : a_{i\mathbf{k}(i)}, c_j \in L^\infty(Q), \|a_{i\mathbf{k}(i)}\|_{L^\infty}, \|c_j\|_{L^\infty} \le M;$$

$$a_{i\mathbf{k}(i)} = 0 \text{ in } Q \setminus Q_{\omega_i}, \text{ and } |a_{i\mathbf{k}(i)}| \ge \delta \text{ on } Q_{\underline{\omega}_i}, \forall i \in \overline{1, n} \Big\}.$$

The main result concerning controllability with one control for linear parabolic systems with tree-like couplings is the following:

THEOREM 3.5. *Consider system (0.31) with coefficients in $\tilde{\mathcal{E}}_{M,\delta,\{\underline{\omega}_i\}_i}$. Then there exists a constant $C = C(M, \delta, \{\underline{\omega}_i\}_i)$ such that for all $z^0 \in H$ there exists $u^* \in L^2(0, T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ which drives in 0 the corresponding solution $z = z^{u^*}$ of (0.31), i.e. $z(T, \cdot) = 0$, and the control satisfies the norm estimate*

$$(0.38) \qquad \|u^* e^{-s\overline{\alpha}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C\|z^0\|_{L^2(\Omega)}.$$

Controllability of nonlinear semilinear parabolic systems with tree-like couplings may be studied in analogy to the star-like case. For this, consider systems of the form

$$(0.39) \qquad \begin{cases} D_t y_0 - \Delta y_0 = g_0(x) + f_0(x, y_0) + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\ D_t y_i - \Delta y_i = g_i(x) + f_i(x, y_{\mathbf{k}(i)}, y_i), i \in \overline{1, n}, & \text{in } (0, T) \times \Omega, \\ y_0 = ... = y_n = 0, & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, & \text{in } \Omega, \end{cases}$$

where $g_j \in L^\infty(\Omega)$, $j \in \overline{0, n}$ and $\overline{y} = (\overline{y}_0, ..., \overline{y}_n) \in [L^\infty(\Omega)]^{n+1}$ is a corresponding stationary solution. We assume the following hypotheses on the nonlinearities:

(H1') $f_0 \in C^1(\Omega \times \mathbb{R}), f_i \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}), i \in \overline{1, n}$ there exist $\omega_1, ... \omega_n \subset \Omega$ open nonempty subsets of Ω satisfying (0.33), (0.34) and

$$(0.40) \qquad (\omega_i \cap \omega_{\mathbf{k}(i)}) \setminus \bigcup_{j \neq i, \mathbf{k}(j) = \mathbf{k}(i)} \omega_j \neq \emptyset, \forall i \in \overline{1, n},$$

and for all $i \in \overline{1, n}$ we have

$$(0.41) \qquad f_i(x, \tau, \xi) = 0 \,\forall x \in \Omega \setminus \omega_i, \, \tau, \xi \in \mathbb{R};$$

(H2') For a family of subdomains $\{\underline{\omega}_i\}_i$ satisfying (0.35), (0.36), by defining for $i \in \overline{1, n}$ the coefficients

$$a^0_{i\mathbf{k}(i)}(x) := \frac{\partial f_i}{\partial y_{\mathbf{k}(i)}}(x, \overline{y}_{\mathbf{k}(i)}(x), \overline{y}_i(x))$$

$$c^0_0(x) := \frac{\partial f_0}{\partial y_0}(x, \overline{y}_0(x)), \, c^0_i(x) := \frac{\partial f_i}{\partial y_i}(x, \overline{y}_{\mathbf{k}(i)}(x), \overline{y}_i(x)),$$

we assume that for some $M_0, \delta_0 > 0$ we have

$$(0.42) \qquad \{a^0_{i\mathbf{k}(i)}, c^0_j\}_{i \in \overline{1,n}, j \in \overline{0,n}} \in \mathcal{E}_{M_0, \delta_0, \{\underline{\omega}_i\}_i, \mathbf{k}}.$$

THEOREM 3.6. *Suppose $\overline{y}$ is a stationary state to uncontrolled problem (0.39) and that functions $f_j, j \in \overline{0, n}$ satisfy hypotheses (H1'), (H2'). Then, for all $\beta_0 > 0$ there exist $\zeta_0 = \zeta_0(\beta_0) > 0$ and $C = C(\beta_0, \{\underline{\omega}_i\}_i, \overline{y})$ such that if $\|y^0 - \overline{y}\|_{L^\infty(\Omega)} < \zeta_0$ there exists a control $u \in L^\infty(Q_{\omega_0})$ satisfying*

$$\|u\|_{L^\infty(Q_{\omega_0})} \leq C\|y^0 - \overline{y}\|_{L^\infty(\Omega)}$$

and the corresponding solution $y = y^u$ to (0.39) verifies

$$y^u(T, \cdot) = \overline{y}, \text{ with } \|y^u(t, \cdot) - \overline{y}\|_{L^\infty(\Omega)} \leq \beta_0, \, t \in [0, T].$$

Part 3. Inverse source problems for parabolic systems

Chapter 4: Stability in L^q-norm for inverse source parabolic problems. We consider systems of linear parabolic equations in bounded subdomains of $\mathbb{R}^N$, coupled in zero and first order terms. The question we address is the Lipschitz stability in L^q-norms, $2 \leq q \leq \infty$ for the source, using observations on the solution in a subdomain. Our result is in the spirit of the results obtained by O. Yu. Imanuvilov and M. Yamamoto in [**61**] for one linear parabolic equation in L^2 norm. We also treat in this chapter the question of partial observations in stability problems, meaning observations on a reduced number of components of the solution. The main tool in our approach is a class of global Carleman estimates. The first result in this chapter is a family of L^q estimates, with general weights, for nonhomogeneous systems of parabolic equations. These are derived through a bootstrap argument, relying on the regularizing effect of parabolic flows, and based on a family of Carleman estimates in L^2 norms with general weights. We use the result in [**61**] and the family of Carleman L^q inequalities to obtain L^q estimates, $q \in [2, +\infty)$, for the inverse source problem. Based on the L^q result and using the approach in [**61**] which uses Carleman estimates for the parabolic system obtained by derivation, we get L^∞ source estimates. Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ a bounded domain with C^2 boundary $\partial\Omega$, $T > 0$ and $Q := (0, T) \times \Omega$. Consider a fixed $\theta \in (0, T)$ to be the observation instant of time and with no loss of generality can be taken $\frac{T}{2}$.

We consider a system of n linear parabolic equations coupled in zero and first order terms:

$$(0.43) \qquad \begin{cases} D_t y_i + L_i y_i + L_i^1 y + L_i^0 y = g_i, & \text{in } (0, T) \times \Omega, \quad i = \overline{1, n} \\ y_i = 0, & \text{on } (0, T) \times \partial\Omega, \quad i = \overline{1, n} \end{cases}$$

where $y := (y_1, ..., y_n)^\top$ and $\{L_i\}_{i=\overline{1,n}}$ is a family of uniformly elliptic operators of second order in divergence form

$$(0.44) \qquad L_i y_i := -\sum_{j,k=1}^{N} D_j(a_i^{jk} D_k y_i) \quad i = \overline{1, n}.$$

The coefficients a_i^{jk} belong to $W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$ and satisfy usual ellipticity condition

$$(0.45) \qquad \sum_{j,k=1}^{N} a_i^{jk}(t, x)\xi_j\xi_k \geq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad (t, x) \in Q, \text{ for some } \mu > 0.$$

The coupling operators are of the form

$$L_i^1 y = \sum_{\substack{k=\overline{1,N} \\ l=\overline{1,n}}} b_i^{kl} D_k y_l, \quad L_i^0 y = \sum_{l=\overline{1,n}} c_i^l y_l, \quad i = \overline{1, n},$$

with coefficients $b_i^{j,k}, c_i^l \in W^{1,\infty}(0, T; L^\infty(\Omega))$.

For $q \geq 2, \tilde{c} > 0, \tilde{\delta} > 0$ and some $\tilde{g} \in [L^{q'}(Q)]^n, \tilde{g} \neq 0$ $(q' = \frac{q}{q-1})$ consider the space of sources for (0.43):

$$(0.46) \qquad \mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{g}} = \left\{ \begin{array}{c} g \in W^{1,1}((0,T); [L^q(\Omega)]^n) : \int_Q g \cdot \tilde{g} \geq \tilde{\delta}\|g\|_{L^q(Q)}, \\ \left| \frac{\partial g(t,x)}{\partial t} \right| \leq \tilde{c}|g(\theta, x)|, a.e.\ (t,x) \in (0,T) \times \Omega \end{array} \right\}.$$

For the system (0.43) we consider the sources $g = (g_i)_{i=\overline{1,n}}$ from the cone $\mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{g}}$. We are interested in obtaining L^q estimates for them in terms of the solution $y = (y_i)_{i=\overline{1,n}}$ measured in $Q_\omega := (0,T) \times \omega$, for some open set $\omega \subset\subset \Omega$. The first result in this chapter is a family of L^q estimates, with general weights, for nonhomogeneous systems of parabolic equations.

We consider a function $\psi \in C^2(\overline{\Omega})$ such that

$$\frac{1}{3} \leq \psi \leq \frac{4}{3}, \quad \psi|_{\partial\Omega} = \frac{1}{3}, \quad \{x \in \overline{\Omega} : |\nabla\psi(x)| = 0\} \subset\subset \omega.$$

One also considers the weight functions

$$(0.47) \qquad \varphi(t,x) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(t,x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

Our result, proved in §**??**, is the following:

THEOREM 4.1. (L^q-Carleman estimate) Let $g \in (L^q(Q))^n$, with $q < \infty$, and $k_0 \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(q, k_0)$, $s_0 = s_0(q, k_0)$, $C = C(q, k_0)$ and $m = m(q)$ such that, for any $\lambda \geq \lambda_0$, $s \geq s_0$ and y a solution to (0.43), the following inequality holds:

$$(0.48) \begin{aligned} &\|\varphi^{k_0-2m-1}ye^{s\alpha}\|_{L^q(Q)} + \|(s\lambda)^{-1}\varphi^{k_0-2m-2}Dye^{s\alpha}\|_{L^q(Q)} \\ &+ \|(s\lambda)^{-2}\varphi^{k_0-2m-4}D_t ye^{s\alpha}\|_{L^q(Q)} + (s\lambda)^{-2}\|\varphi^{k_0-2m-4}D^2 ye^{s\alpha}\|_{L^q(Q)} \\ &\leq C\left[(s\lambda)^{2m}\|\varphi^{k_0}ye^{s\alpha}\|_{L^2(Q_\omega)} + (s\lambda)^{2m-\frac{3}{2}}\|\varphi^{k_0+3}ge^{s\alpha}\|_{L^q(Q)} \right]. \end{aligned}$$

If $N + 1 < q < \infty$ we have a C^γ estimate for y, with $\gamma = 1 - (N+1)/q$:

$$(0.49) \begin{aligned} &\|\varphi^{k_0-2m-1}ye^{s\alpha}\|_{C^\gamma(Q)} \leq \\ &\leq C\left[(s\lambda)^{2m+2}\|\varphi^{k_0+2}ye^{s\alpha}\|_{L^2(Q_\omega)} + (s\lambda)^{2m+\frac{1}{2}}\|\varphi^{k_0+5}ge^{s\alpha}\|_{L^q(Q)} \right]. \end{aligned}$$

The main result regarding the inverse source problem is obtaining an estimate on the source $g = (g_1, ..., g_n)^\top$ using data on the solution measured on a subdomain, $\omega \subset \Omega$. The first such result is the following and concerns estimates of the source in L^q norm $q \geq 2$:

THEOREM 4.2. Let $q \in [2, \infty)$, $\tilde{c} > 0$, $\tilde{\delta} > 0$ and $\tilde{g} \in [L^{q'}(Q)]^n, \tilde{g} \neq 0$. Then there exists a constant $C = C(q, \tilde{c}, \tilde{\delta}, \tilde{g}) > 0$ such that for sources $g \in \mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{g}}$ of (0.43) and corresponding solutions $y \in L^q\left(0, T; (W_0^{1,q} \cap W^{2,q}(\Omega))^n\right)$ the following estimate holds:

$$(0.50) \qquad \|g\|_{L^q(Q)} \leq C\left(\|y\|_{L^q(Q_\omega)} + \|y(\theta, \cdot)\|_{W^{2,q}(\Omega)} \right).$$

For stability estimates of the source in L^∞ norm we assume that the sources are more regular, belonging to $C^\gamma(Q)$ for some $\gamma \in (0,1)$. In fact, we need only $g(\theta, \cdot) \in C^\gamma(\Omega)$.

THEOREM 4.3. *Suppose that the coefficients of the operators L_i, L_i^1, L_i^0 are smooth in $\overline{Q}$. Let $\gamma \in (0,1)$ and consider sources in* (0.43) *with Hölder regularity $g \in (C^\gamma(Q))^n \cap \mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{g}}$, $q = \frac{N+1}{1-\gamma}$. Then there exists $C = C(\gamma, \tilde{c}, \tilde{\delta}, \tilde{g}) > 0$, such that*

$$(0.51) \qquad \|g\|_{L^\infty(Q)} \leq C \left(\|y\|_{L^q(Q_\omega)} + \|y(\theta, \cdot)\|_{C^{2+\gamma}(\Omega)} \right).$$

Chapter 5: Stability in inverse source problems for nonlinear reaction-diffusion systems. We consider systems of semilinear parabolic equations, coupled in zero order terms, and we study an inverse problem addressing the question of source estimation in L^q and L^∞ norms in terms of norms of the solution measured in a subdomain. The systems we study arise from reaction-diffusion models, physical phenomena like heat transfer, population dynamics, etc. In this context the sources have positive entries and also the solutions remain in the cone of positive functions as some extra hypotheses on the nonlinear part, related to parabolic maximum principle, are assumed.

Our result has as starting point the work of O. Yu. Imanuvilov and M. Yamamoto, [61], where the authors have considered linear parabolic equations in bounded domains and established L^2 estimates for the source. In this chapter we improve the result to the more general case of L^q, respectively L^∞ estimates for the source, in a linearized model, and apply these results to nonlinear models of reaction-diffusion systems. We are able to obtain a sharper source estimate, without involving the time derivative of the solution in the right side of the estimates and the method uses a family of Carleman estimates with generalized weights and an argument based on the maximum principle for coupled parabolic systems.

We use the L^2 Carleman estimates as the start point to a bootstrap procedure, which leads to a corresponding class of $L^q, q \geq 2$ Carleman estimates with independent parameters and generalized weights of exponential type for nonhomogeneous parabolic systems with various homogeneous boundary conditions. The bootstrap argument is based on the regularizing effect of the heat flow in L^p spaces (see, for example,the monograph of O. A. Ladyzenskaja, V. A. Solonikov, N. N. Ural'ceva, [64]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\omega \subset\subset \Omega$ be an open nonempty subset of Ω, $T > 0$ and $Q = (0,T) \times \Omega$.

We denote by $(L_i)_{i=\overline{1,n}}$ a family of n uniformly elliptic operators of second order in divergence form

$$(0.52) \qquad L_i w = - \sum_{j,k=1}^{N} D_j(a_i^{jk} D_k w)$$

with coefficients $a_i^{jk} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, $i = \overline{1,n}, j, k = \overline{1,N}$. Denote by $A_i = (a_i^{jk})_{j,k=\overline{1,N}}$ the matrix of coefficients in principal part which we assume satisfying the

usual uniform ellipticity condition

$$(0.53) \qquad \exists \mu > 0 \ s.t. \ \sum_{j,k=1}^{N} a_i^{jk}(t,x)\xi_j\xi_k \geq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad (t,x) \in Q, i = \overline{1,n}.$$

Consider also the first order operators (w is considered a scalar function),

$$(0.54) \qquad L_i^1 w = \sum_{k=1}^{N} b_i^k D_k w, \ i = \overline{1,n},$$

with coefficients $b_i^k \in W^{1,\infty}(0,T; L^\infty(\Omega))$.

We study the following reaction-diffusion system of n coupled parabolic equations

$$(0.55) \qquad \begin{cases} D_t y_i - \sum_{j,k=1}^{N} D_j(a_i^{jk} D_k y_i) + L_i^1 y_i + f_i(y_1,...,y_n) = g_i, & (0,T) \times \Omega, \\ \beta_i(x)\dfrac{\partial y_i}{\partial n_{A_i}} + \eta_i(x)y_i = 0, & (0,T) \times \partial\Omega, \end{cases}$$

where $g_i \geq 0, i = \overline{1,n}$ are the internal sources acting in each equation of the system. In the following, when referring to a vector function $g = (g_i)_{i\in\overline{1,n}}^\top$ to be positive, like $g \geq 0$, we consider the inequality satisfied on each component of the vector, $g_i \geq 0, i = \overline{1,n}$.

In the boundary conditions, we denoted by $\frac{\partial}{\partial n_{A_i}}$ the conormal derivatives, $\frac{\partial y}{\partial n_{A_i}} = \langle A_i \nabla y, n \rangle$, $A_i = (a_i^{jk})_{j,k}$. We impose that $\beta_i, \eta_i \in C^2(\partial\Omega)$ such that

$$(0.56) \qquad \beta_i > 0 \text{ on } \partial\Omega \qquad \text{or} \qquad \beta_i \equiv 0 \text{ and } \eta_i \equiv 1 \text{ on } \partial\Omega.$$

The coupling is given through the C^1 nonlinearities $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $f_i(0) = 0, i = \overline{1,n}$ and we introduce the following hypotheses:

(H1) (*quasimonotonicity*) for some $\varepsilon_0 > 0$, $\frac{\partial f_i}{\partial y_j}(y_1,\ldots,y_n) \leq 0$, $y \in \mathcal{V}_{\varepsilon_0}(0) := \{y \geq 0, \|y\| \leq \varepsilon_0\}, j \neq i, i,j = 1,\ldots,n$;

(H2) $f_i(y_1,\ldots,y_{i-1},0,y_{i+1},\ldots,y_n) \leq 0, i = \overline{1,n}, y \geq 0$.

In the following we consider a fixed instant of time $\theta \in (0,T)$ which can be chosen, for the ease of computations $\theta = \frac{T}{2}$.

In order to describe the framework of our problem, we introduce the following sets of functions (sources and corresponding solutions).
Let $\tilde{G}$ be a compact subset of $[L^{q'}(Q)]^n$ with $q' = \frac{q}{q-1}$ such that $0 \notin \tilde{G}$. For $q \geq 2, \tilde{c} > 0$, $\tilde{\delta} > 0$ consider the sets of sources:

$$(0.57) \qquad \mathcal{G}_{q,\tilde{\delta},\tilde{G}} = \left\{ \begin{array}{l} g \in W^{1,1}((0,T); [L^q(\Omega)]^n) : g \geq 0 \\ \text{and } \exists \tilde{g} \in \tilde{G} \text{ s.t. } \int_Q g \cdot \tilde{g}\,dx\,dt \geq \tilde{\delta}\|g\|_{L^q(Q)} \end{array} \right\}$$

and

$$(0.58) \qquad \mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{G}} = \left\{ \begin{array}{l} g \in W^{1,1}((0,T); [L^q(\Omega)]^n) : g \geq 0, \\ \left|\dfrac{\partial g(t,x)}{\partial t}\right| \leq \tilde{c}|g(\theta,x)|, \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \\ \text{and } \exists \tilde{g} \in \tilde{G} \text{ s.t. } \int_Q g \cdot \tilde{g}\,dx\,dt \geq \tilde{\delta}\|g\|_{L^q(Q)} \end{array} \right\}.$$

Also, consider the set of functions,

$$(0.59) \qquad \mathcal{F}_{q,M} = \{y \in [W_q^{2,1}(Q) \cap L^\infty(Q)]^n : y \geq 0, \|y\|_{L^\infty(Q)} \leq M\}.$$

The main results concerning the stability for nonlinear parabolic systems are the following two theorems:

THEOREM 5.1. (L^q stability estimates) *Let* $2 \leq q < \infty$. *Let* $\tilde{\delta} > 0, M > 0$, *a compact set* $\tilde{G} \subset L^{q'}(Q)$, $0 \notin \tilde{G}$ *and assume that the sources in* (0.55) *belong to* $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ *and the associated solutions satisfy* $y \in \mathcal{F}_{q,M}$. *Assume also that one of the following conditions,* (A) *or* (B), *concerning nonlinearity* f, *holds:*

(A) f *satisfies the hypothesis* (H1) *in the whole cone* $y \geq 0$ *and* (H2),

or

(B) $q > \frac{N+2}{2}$ *and* f *satisfies hypotheses* (H1), (H2).

Then an L^q *stability estimate holds: there exists* $C = C(q, \tilde{\delta}, M, \tilde{G}) > 0$ *such that*

$$(0.60) \qquad\qquad \|g\|_{L^q(Q)} \leq C\|y\|_{L^q(Q_\omega)}.$$

THEOREM 5.2. (L^∞ stability estimates) *Let* $\varrho \in (0,1)$, $q = \frac{N+1}{1-\varrho}$ *and* $\theta \in (0,T)$ *an intermediate observation instant of time. Consider* $\tilde{\delta} > 0$, $M > 0$ *and a compact set* $\tilde{G} \subset L^{q'}(Q)$, $0 \notin \tilde{G}$ *such that the sources in* (0.55) *belong to* $\mathcal{G}_{q,\tilde{c},\tilde{\delta},\tilde{G}} \cap C^\varrho(Q)$ *and the associated solutions* $y \in \mathcal{F}_{q,M}$. *Assume also that one of the conditions* (A) *or* (B) *holds.*

Then there exists $C = C(\varrho, \tilde{c}, \tilde{\delta}, M, \tilde{G}) > 0$ *such that an* L^∞ *source estimate holds:*

$$(0.61) \qquad\qquad \|g\|_{L^\infty(Q)} \leq C(\|y\|_{L^q(Q_\omega)} + \|y(\theta, \cdot)\|_{C^{2+\varrho}(\Omega)}).$$

The approach for obtaining source estimates for nonlinear systems is combining a priori estimates for the solution with source estimates for associated linear systems which in a certain sense approximate the nonlinear model. The results in the linear case give informations on the source in the nonlinear problem under apriori L^∞ bounds of the solutions.

Consequently, for the beginning we consider a generic linear parabolic problem, with the same principal part as the nonlinear system, with one of the homogeneous boundary conditions (Dirichlet, Neumann or Robin) on each component of the vector solution (0.55),

$$(0.62) \qquad \begin{cases} D_t y_i + L_i y_i + L_i^1 y_i + L_i^0 y = g_i, & (0,T) \times \Omega, \\ \beta_i(x)\frac{\partial y_i}{\partial n_{A_i}} + \eta_i(x)y_i = 0, & (0,T) \times \partial\Omega, \end{cases} \quad i = \overline{1,n}$$

where $g_i \geq 0, i = \overline{1,n}$ are the internal sources and β_i, η_i are given as before in (0.56).

The lower-order operators are given by (w is a scalar function, y is vector valued function):

$$(0.63) \qquad L_i^1 w = \sum_{k=\overline{1,N}} b_i^k D_k w, \quad L_i^0 y = \sum_{l=\overline{1,n}} c_i^l y_l, \quad i = \overline{1,n},$$

with coefficients $b_i^k, c_i^l \in W^{1,\infty}(0,T;L^\infty(\Omega))$, and the coupling is done only through the zero-order terms $c_i^l \leq 0$, $i \neq l$, $i,l \in \overline{1,n}$.

We are interested in obtaining L^q and L^∞ estimates for the source $g = (g_i)_{i=\overline{1,n}} \in \mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ in terms of the solution y measured in Q_ω. The result in the linear case is the following:

THEOREM 5.3. *Let* $2 \leq q < \infty, \tilde{\delta} > 0, \tilde{c} > 0$ *and a compact set* $\tilde{G} \subset L^{q'}(Q)$, $0 \notin \tilde{G}$. *Then, for sources* g *in* $\mathcal{G}_{q,\tilde{\delta},\tilde{G}}$ *and corresponding solutions* $y \geq 0$ *to* (0.62) *belonging to* $[W_q^{2,1}(Q)]^n$, *there exists* $C = C(q, \tilde{\delta}, \tilde{G}) > 0$, *such that*

$$(0.64) \qquad \qquad \|g\|_{L^q(Q)} \leq C\|y\|_{L^q(Q_\omega)}.$$

Moreover, for fixed $\theta \in (0, T)$, *given* $\varrho \in (0, 1)$, *choosing* $q = \dfrac{N+1}{1-\varrho}$ *and considering sources* g *in* $\mathcal{G}_{q,\tilde{\delta},\tilde{c},\tilde{G}} \cap C^\varrho(Q)$ *with corresponding solutions* $y \geq 0$ *to* (0.62) *belonging to* $[W_q^{2,1}(Q)]^n$, *there exists* $C = C(\varrho, \tilde{\delta}, \tilde{c}, \tilde{G}) > 0$, *such that*

$$(0.65) \qquad \qquad \|g\|_{L^\infty(Q)} \leq C(\|y\|_{L^q(Q_\omega)} + \|y(\theta, \cdot)\|_{C^{2+\varrho}(\Omega)}).$$

The proof of the above theorem relies on L^q Carleman estimates for the parabolic systems under homogeneous boundary conditions (Dirichlet, Neumann or Robin) and an argument based on the Maximum Principle for systems of parabolic equations.

Consider now weakly coupled linear systems of form (0.62) where the boundary operator is given by

$$\mathcal{B}y = (B_i y_i)_{i=\overline{1,n}}, \mathcal{B}_i y_i = \beta_i(x)\frac{\partial y_i}{\partial n_{A_i}} + \eta_i(x)y_i, i = \overline{1,n}.$$

Under the assumed hypothesis that the off-diagonal terms of the matrix L^0 are non-positive,

$$(0.66) \qquad \qquad c_i^l \leq 0, \ i \neq l, \ i, l \in \overline{1,n},$$

the results from [87], [3] give that if $y_i(0, \cdot) \geq 0$ in Ω then we have $y_i \geq 0$ in the whole domain $(0, T) \times \Omega$. Moreover, if the solution is zero at an interior point $(t_0, x_0) \in (0, T) \times \Omega$ then $y \equiv 0$ for all $t < t_0$.

The main result concerning the L^q Carleman estimates for systems of linear parabolic equations (0.62), that we prove in §** uses some auxiliary functions. Consider an open subset $\omega \subset\subset \Omega$ and a function $\psi \in C^2(\overline{\Omega})$ such that

$$\frac{1}{3} \leq \psi \leq \frac{4}{3}, \quad \psi|_{\partial\Omega} = \frac{1}{3}, \quad \{x \in \overline{\Omega} : |\nabla\psi(x)| = 0\} \subset\subset \omega.$$

One also considers the weight functions

$$(0.67) \qquad \varphi(t, x) := \frac{e^{\lambda\psi(x)}}{t(T - t)}, \quad \alpha(t, x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T - t)}.$$

The result concerning the L^q Carleman estimates for systems of linear parabolic equations (0.62) is the following

PROPOSITION 5.1. (L^q-*Carleman estimate*) *Let* $g \in (L^q(Q))^n$, *with* $2 \leq q < \infty$. *Then there exist* $s_0 = s_0(q), \lambda_0 = \lambda_0(q)$, *such that if* $\lambda > \lambda_0, s', s > s_0, \frac{s'}{s} > \Gamma > 1$, *then there exists* $C = C(q, \Gamma)$ *such that the solutions* $y \in W_q^{2,1}(Q)$ *to* (0.62), *satisfy the*

estimate:

$$(0.68) \quad \begin{aligned} &\|ye^{s'\alpha}\|_{L^q(Q)} + \|(Dy)e^{s'\alpha}\|_{L^q(Q)} + \|(D^2 y)e^{s'\alpha}\|_{L^q(Q)} + \|(D_t y)e^{s'\alpha}\|_{L^q(Q)} \\ &\leq C \left[\|ge^{s\alpha}\|_{L^q(Q)} + \|ye^{s\alpha}\|_{L^2(Q_\omega)} \right]. \end{aligned}$$

Part 4. Regularity and Carleman estimates in $L^q(L^p)$ spaces for parabolic problems

Chapter 6: On the parabolic regularity, Sobolev embeddings and global Carleman estimates in $L^q(L^p)$ spaces. We present some parabolic regularity results that may be derived from existing theory in the cited literature. We chose to present it in a more concentrated appearance, which is useful for studying regularity in nonlinear parabolic problems, through bootstrap arguments, when the nonlinearity depends on the state itself y and its first order derivatives, Dy. The parabolic regularity results are then used to present a clear proof to classical embeddings for anisotropic Sobolev spaces and we also use this approach to Sobolev embeddings of $W_{p,q}^{2,1}(Q)$ spaces. We discuss Gagliardo-Nirenberg type inequalities for anisotropic Sobolev spaces. We also apply our regularity arguments to establish global Carleman parabolic estimates in $L^q(L^p)$, $q, p > 2$ spaces, for nonhomogeneous parabolic equations.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and denote by $Q = (0, T) \times \Omega$. We consider parabolic problems of the form

$$(0.69) \quad \begin{cases} D_t y(t, x) + Ly(t, x) = f(t, x) & t \in (0, T), x \in \Omega, \\ y(t, x) = 0 & t \in (0, T), x \in \partial\Omega, \\ y(0, x) = y_0(x) & x \in \Omega, \end{cases}$$

where L is an uniformly elliptic operator of the form

$$(0.70) \quad Ly = -\sum_{j,k=1}^n D_j(a_{jk} D_k y) + \sum_{k=1}^n b_k D_k y + cy.$$

The coefficients satisfy the regularity assumptions $a_{jk} \in W^{1,\infty}(\Omega)$, $b^k, c \in L^\infty(\Omega)$ and those in principal part satisfy for some $\mu > 0$ the ellipticity condition

$$(0.71) \quad \sum_{j,k=1}^n a_{jk}(x)\xi_j \xi_k \geq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \, x \in \Omega.$$

For $p, q \in [1, \infty)$ consider the spaces (see [102, 103]):

$$W_{p,q}^{2,1}(Q) = L^q(W^{2,p}(\Omega)) \cap W^{1,q}(L^p(\Omega)).$$

One of the main results in the chapter is about Sobolev type embeddings for $W_{p,q}^{2,1}(Q)$, and the approach will rely on the regularity of flows generated by analytic semigroups. The L^p realization for some $p \in (1, \infty)$, with homogeneous Dirichlet boundary conditions for L takes into account the L^p regularity theory for elliptic equations (see [58]) and is defined as $A = A_p : D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $Au = Lu$, $u \in D(A)$.

Without loss of generality concerning regularity we may suppose that L is positive. Moreover, one knows that $-A$ generates an analytic semigroup in L^p. Maximum principle

applied to elliptic operator L shows that $(\lambda I + A)^{-1}$ is positivity preserving and by Theorem 6.4 we find that $A = A_p$ has bounded imaginary powers. We have thus:

THEOREM 6.5. *The operator $A = A_p$ with $p \in (1, \infty)$, which is the L^p realization of elliptic operator L with homogeneous boundary conditions on $\partial\Omega$, has the property that, for $\gamma \in (0, 1)$,*

$$D(A^\gamma) = [L^p(\Omega), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]_\gamma.$$

($[X, Y]_\gamma$ denotes the complex interpolation space between Banach spaces X and Y.)

Relation between domains of fractional powers of operator A and Sobolev-Slobodeckii spaces is recalled in the next proposition:

PROPOSITION 6.1. *Consider $\gamma \in (0, 1)$. Then, if $p \geq 2$, one has the continuous embeddings*

$$D(A^\gamma) \subset H^{2\gamma,p}(\Omega) \subset W^{2\gamma,p}(\Omega).$$

If $1 < p < 2$ and $\gamma' < \gamma$,

$$D(A^\gamma) \subset H^{2\gamma,p}(\Omega) \subset W^{2\gamma',p}(\Omega).$$

with continuous embeddings.

Consider $X = L^p(\Omega)$ and the parabolic problem with homogeneous initial data:

(0.72) $$y' + Ay = f, y(0) = 0, t \in (0, T)$$

with A the L^p realization of parabolic operator L with Dirichlet boundary conditions. It turns out that $D(A) = W^{2,p} \cap W_0^{1,p}(\Omega)$. The mild solution is given by

(0.73) $$y(t) = \int_0^t e^{-(t-s)A} f(s) ds.$$

Our purpose is to obtain regularity in $L^r(D(A^\gamma))$ and, subsequently, relating $D(A^\gamma)$ to Bessel potential and Sobolev-Slobodeckii spaces, in $L^r(H^{s,p}(\Omega))$ and $L^r(W^{s,p}(\Omega))$, for some $s > 0, r > 1$.

PROPOSITION 6.2. *Consider $q, p \in (1, \infty)$ and $f \in L^q(L^p(\Omega))$. For $r \in (q, \infty]$ and $\theta = 2 + \frac{2}{r} - \frac{2}{q}$, the mild solution y to (0.72), given by (0.73), satisfies the regularity estimate:*

(0.74) $$\|y\|_{L^r(H^{\theta,p}(\Omega)))} \leq C\|f\|_{L^q(L^p(\Omega))},$$

with a constant $C = C(p, q, r)$.

Moreover, for $r_1 \in (q, \infty)$ if $q \geq 2$ and $r_1 \in \left(q, \frac{2q}{2-q}\right]$ if $q \in (1, 2)$, and choosing $\tilde{\theta} = 1 + \frac{2}{r_1} - \frac{2}{q}$, the gradient of the mild solution y satisfies the regularity estimate:

(0.75) $$\|Dy\|_{L^{r_1}(H^{\tilde{\theta},p}(\Omega))} \leq \tilde{C}\|f\|_{L^q(L^p(\Omega))}.$$

with a constant $\tilde{C} = \tilde{C}(p, q, r_1)$.

COROLLARY 6.1. *With $r \in (q, \infty)$ and $\theta = 2 + \frac{2}{r} - \frac{2}{q}$ we have the estimates:*

- For $\theta p \leq n$, choosing $\tilde{p} \leq \frac{np}{n-\theta p}$ if $\theta p < n$ and choosing arbitrarily $\tilde{p} \in [p, \infty)$ if $\theta p = n$, one has

$$\|y\|_{L^r(L^{\tilde{p}}(\Omega)))} \leq C(p, q, r, \tilde{p})\|f\|_{L^q(L^p(\Omega))};$$

- If $\theta p > n$, then $y \in L^r(C^{k+\alpha}(\overline{\Omega}))$ with $\alpha \in (0, 1], k \in \{0, 1\}, k + \alpha = \theta - \frac{n}{p}$ and

$$\|y\|_{L^r(C^{k+\alpha}(\overline{\Omega}))} \leq C(p, q, r, \tilde{p})\|f\|_{L^q(L^p(\Omega))}.$$

Moreover, for $r_1 \in (q, \infty)$ if $q \geq 2$ and $r_1 \in (q, \frac{2q}{2-q}]$ if $q \in (1, 2)$, denoting by $\tilde{\theta} = 1 + \frac{2}{r_1} - \frac{2}{q}$ we have the following estimates for the gradient of the solution:

- For $\tilde{\theta}p \leq n$, choosing $\tilde{p}_1 \leq \frac{np}{n-\tilde{\theta}p}$ if $\tilde{\theta}p < n$ and choosing arbitrarily $\tilde{p}_1 \in [p, \infty)$ if $\tilde{\theta}p = n$, one has

$$\|Dy\|_{L^{r_1}(L^{\tilde{p}_1}(\Omega)))} \leq C(p, q, r_1, \tilde{p}_1)\|f\|_{L^q(L^p(\Omega))};$$

- If $\tilde{\theta}p > n$, then $y \in L^{r_1}(C^{\alpha_1}(\overline{\Omega}))$ with $\alpha_1 \in (0, 1), \alpha_1 = \tilde{\theta} - \frac{n}{p}$ and

$$\|Dy\|_{L^{r_1}(C^{\alpha_1}(\Omega)))} \leq C(p, q, r_1, \tilde{p})\|f\|_{L^q(L^p(\Omega))}.$$

Concerning Sobolev embeddings for $W_{p,q}^{2,1}(Q)$ spaces we obtain the following result:

THEOREM 6.8. *Consider* $u \in W_{p,q}^{2,1}(Q), p, q \in (1, \infty)$.
Then $u \in Z_1$ *where*

$$Z_1 = \begin{cases} L^r(L^{\tilde{p}}(\Omega)), r \in [q, \infty], \tilde{p} \leq \frac{np}{n-(2+\frac{2}{r}-\frac{2}{q})p}, & \text{if } (2 + \frac{2}{r} - \frac{2}{q})p < n, \\ L^r(L^{\tilde{p}}(\Omega)), r \in [q, \infty], \tilde{p} \in [p, \infty), & \text{if } (2 + \frac{2}{r} - \frac{2}{q})p = n, \\ L^r(C^{k+\alpha}(\overline{\Omega})), \alpha \in (0, 1], k \in \{0, 1\}, & k + \alpha = 2 + \frac{2}{r} - \frac{2}{q} - \frac{n}{p}, \\ & \text{if } (2 + \frac{2}{r} - \frac{2}{q})p > n \end{cases}$$

and there exists $C = C(p, q, r, \tilde{p})$, respectively $C = C(p, q, r)$ in the third case, such that

$$\|u\|_{Z_1} \leq C\|u\|_{W_{p,q}^{2,1}(Q)}.$$

Moreover, $Du \in Z_2$ where

$$Z_2 = \begin{cases} L^{r_1}(L^{\tilde{p}_1}(\Omega)), r_1 \in [q, \infty], \tilde{p}_1 \leq \frac{np}{n-(1+\frac{2}{r_1}-\frac{2}{q})p}, & \text{if } (1 + \frac{2}{r_1} - \frac{2}{q})p < n, \\ L^{r_1}(L^{\tilde{p}_1}(\Omega)), r_1 \in [q, \infty], \tilde{p}_1 \in [p, \infty), & \text{if } (1 + \frac{2}{r_1} - \frac{2}{q})p = n, \\ L^{r_1}(C^{\alpha}(\overline{\Omega})), \alpha \in (0, 1], \alpha = 1 + \frac{2}{r_1} - \frac{2}{q} - \frac{n}{p} & \text{if } (1 + \frac{2}{r_1} - \frac{2}{q})p > n \end{cases}$$

and there exists $C = C(p, q, r_1, \tilde{p}_1)$, respectively $C = C(p, q, r_1)$ in the third case, such that

$$\|Du\|_{Z_2} \leq C\|u\|_{W_{p,q}^{2,1}(Q)}.$$

THEOREM 6.9. *For* $p, q \in (1, \infty)$, *suppose there exists* $\gamma \in (0, \frac{q-1}{q})$ *with* $2\gamma - \frac{n}{p} > 0$ *not an integer. Then the space* $W_{p,q}^{2,1}(Q)$ *is continuously embedded in* $C^{\frac{q-1}{q}-\gamma}(C^{k+\alpha}(\overline{\Omega}))$, *where* $k \in \{0, 1\}, \alpha \in (0, 1), k + \alpha = 2\gamma - \frac{n}{p}$.

One may easily use Theorem **??** to obtain interpolation inequalities of Gagliardo type between spaces $W_{p,q}^{2,1}(Q)$ and $L^\sigma(L^\tau(\Omega))$, with $p, q, \sigma, \tau \in (1, \infty)$. If $W_{p,q}^{2,1}(Q) \subset L^r(L^{\tilde{p}})(\Omega)$ with continuous injections and $u \in W_{p,q}^{2,1}(Q) \cap L^\sigma(L^\tau(\Omega))$, then $u \in [L^r(L^{\tilde{p}}(\Omega)), L^\sigma(L^\tau(\Omega))]_\theta$,

$\theta \in (0,1)$ and satisfies the inequality

$$\|u\|_{L^{\sigma_\theta}(L^{\tau_\theta}(\Omega))} \leq C(\theta, p, q, \sigma, \tau)\|u\|^{1-\theta}_{W^{2,1}_{p,q}(Q)}\|u\|^{\theta}_{L^{\sigma}(L^{\tau}(\Omega))},$$

where $\frac{1}{\sigma_\theta} = \frac{\theta}{\sigma} + \frac{1-\theta}{r}$ and $\frac{1}{\tau_\theta} = \frac{\theta}{\tau} + \frac{1-\theta}{\tilde{p}}$.

Let $\omega \subset\subset \Omega$. One needs (and existence is guaranteed, see [57]) an auxiliary function ψ with the following properties:

$$\psi_0 \in C^2(\overline{\Omega}),\ 0 < \psi_0 \text{ in } \Omega, \quad \psi_0|_{\partial\Omega} = 0, \quad \{x \in \overline{\Omega} : |\nabla\psi_0(x)| = 0\} \subset\subset \omega.$$

Denote by

$$(0.76) \qquad\qquad\qquad\qquad \psi := \psi_0 + K,$$

for a positive constant $K > 0$ which is fixed such that $\frac{\sup \psi}{\inf \psi} < \delta$ small enough (see [55]). Introduce also, for parameters $s, \lambda > 0$ the auxiliary functions:

$$(0.77) \qquad \varphi(t, x) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(t, x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

THEOREM 6.10. *Let* $f \in L^q(L^p(\Omega)), p, q \in [2, \infty)$ *and* $k_0 \in \mathbb{R}$. *Then there exist* $m = m(q, p) \in \mathbb{N},\ \lambda_0 = \lambda_0(p, q, k_0),\ s_0 = s_0(p, q, k_0)$ *and* $C = C(p, q, k_0) > 0$ *such that, for any* $\lambda \geq \lambda_0, s \geq s_0$, *the following inequality holds:*

$$\|\varphi^{k_0-2m}ye^{s\alpha}\|_{L^q(L^p(\Omega))} + s^{-1}\lambda^{-1}\|\varphi^{k_0-2m-1}Dye^{s\alpha}\|_{L^q(L^p(\Omega))}$$

$$(0.78)$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^2(Q_\omega)} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p(\Omega))}]$$

$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^q(L^p(\omega))} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p(\Omega))}].$$

Appendices. The thesis concludes with a set of appendices addressing mathematical framework, fundamental results and techniques on which the the thesis is developed. These appendices are as follows:

In Appendix A we recall definitions and embedding theorems concerning isotropic and anisotropic Sobolev spaces.

The Appendix B deals with fundamental results from C_0-semigroups theory, describing the abstract framework for the study of evolution partial differential equations and, in particular, the parabolic problems in which we are interested.

Appendix C is devoted to maximum principles for parabolic equations and systems; in the latter situation maximum principles are related to invariance properties for parabolic flows. This part comes as a support for the inverse problems we have presented.

Appendix D presents some fixed point theorems which are usually used in exitence for nonlinear problems (existence to Cauchy problems for semilinear equations, exitence of controls in controllability of nonlinear equations, *etc.*).

In Appendix E we present results concerning controllability, observability and Carleman estimates.

Appendix F is dedicated to inverse source problems, more precisely the result from [61], the starting point for our research in this direction.

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