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A Trotter type result for the stochastic porous media equations

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Abstract

This paper proves the continuous dependence with respect to diffusivity of the solutions to the stochastic porous media equations with noncoercive monotone diffusivity function and multiplicative noise.

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1 Introduction

Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^d$ $(1 \leq d \leq 3)$ with smooth boundary $\partial\mathcal{O}$. We also consider the stochastic partial differential equations

$$(1) \qquad \begin{cases} dX(t) - \Delta\Psi(X(t))\,dt \ni \sigma(X(t))\,dW(t), & \text{in} \quad (0,T)\times\mathcal{O} \\ \Psi(X(t)) \ni 0, & \text{on} \quad (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \text{in} \quad \mathcal{O} \end{cases}$$

where x is the initial data and $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone (possibly multivalued) graph with polynomial growth and $\sigma(X)$ is defined by

$$(2) \qquad \sigma(x)\,h = \sum_{k=1}^{\infty} \mu_k (h, e_k)\, x e_k, \qquad \forall \;\; x \in H^{-1}(\mathcal{O}), \quad \forall \;\; h \in L^2(\mathcal{O}),$$

where $(.,.)$ is the scalar product in $L^2(\mathcal{O})$.

We note that

$$\sigma(X)\,dW = \sum_{k=1}^{\infty} \mu_k X d\beta_k e_k, \qquad \forall \quad t \geq 0,$$

which is linear in X. Here $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$, $\{\mu_k\}$ is a sequence of positive numbers and $\{\beta_k\}$ a sequence of independent standard Brownian motion on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$.

In this work we shall suppose that the sequence $\{\mu_k\}$ is such that

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 = C < \infty, \tag{3}$$

where λ_k are the eigenvalues of the Laplace operator $-\Delta$ in $\mathcal{O}$ with Dirichlet boundary conditions.

Recall that the operator $A : D(A) \subset H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$ is defined by $Ax = -\Delta \Psi(x)$ where

$$D(A) = \left\{ x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \Psi(x) \in H_0^1(\mathcal{O}) \right\}.$$

The Sobolev space $H^{-1}(\mathcal{O})$ (the dual of $H_0^1(\mathcal{O})$) is endowed with the norm

$$|x|_{H^{-1}(\mathcal{O})} = |x|_{-1} = \left| (-\Delta)^{-1} x \right|_{H_0^1(\mathcal{O})}.$$

(Here $(-\Delta)^{-1} x = y$ is the solution to Dirichlet problem $-\Delta y = x$ in $\mathcal{O},\ y \in H_0^1(\mathcal{O})$).

The scalar product in $H^{-1}(\mathcal{O})$ is given by

$$\langle x, z \rangle_{-1} = \int_{\mathcal{O}} (-\Delta)^{-1} x z d\xi, \qquad \forall x, z \in H_0^1(\mathcal{O}).$$

We note that since $d \leq 3$ we have by Sobolev embedding theorem

$$|e_k|_\infty \leq C |e_k|_{H^2(\mathcal{O})} \leq C |\Delta e_k|_{L^2(\mathcal{O})} \leq C \lambda_k$$

and for some constant $c_1 > 0$

$$\sum_{k=1}^{\infty} \mu_k^2 |x e_k|_{-1}^2 \leq c_1 \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 |x|_{-1}^2 \leq C_1 |x|_{-1}^2, \quad \forall \ x \in H^{-1}(\mathcal{O}).$$

We obtain that $\sigma(x)$ is a Hilbert Schmidt from $L^2(\mathcal{O})$ to $H^{-1}(\mathcal{O})$. Note that since σ is linear we have that $x \to \sigma(x)$ is Lipschitz from $H^{-1}(\mathcal{O})$ to $L_2\left(L^2(\mathcal{O}), H^{-1}(\mathcal{O})\right)$.

Recall from [9] the following definition:

Definition 1 *Let $x \in H^{-1}(\mathcal{O})$. An $H^{-1}(\mathcal{O})$ valued continuous $\mathcal{F}_t-$ adapted process $X = X(t, x)$ is called a solution to (1) on $[0, T]$ if*

$$X \in L^p(\Omega \times (0, T) \times \mathcal{O}) \cap L^2\left(0, T; L^2\left(\Omega, H^{-1}(\mathcal{O})\right)\right)$$

and there exists $\eta \in L^{p/m}(\Omega \times (0, T) \times \mathcal{O})$ such that $\mathbb{P}-$a.s.

(4)

$$\langle X(t), e_j \rangle_2 = \langle x, e_j \rangle_2 + \int_0^t \int_{\mathcal{O}} \eta(s, \xi) \Delta e_j(\xi) d\xi ds$$
$$+ \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s) e_k, e_j \rangle_2 d\beta_k(s), \quad \forall\ j \in \mathbb{N}, \quad t \in [0, T],$$

and

$$\eta \in \Psi(X), \quad a.e. \quad in \quad \Omega \times (0, T) \times \mathcal{O}.$$

Here m is the exponent arising in the assumption (6) and $\{e_k\}$ is the above orthonormal basis in $L^2(\mathcal{O})$. Taking into account that $-\Delta e_k = \lambda e_k$ in $\mathcal{O}$ we may equivalently write (4) as follows

$$\langle X(t), e_j\rangle_{-1} = \langle x, e_j\rangle_2 - \int_0^t \int_{\mathcal{O}} \eta(s,\xi)\, e_j(\xi)\, d\xi ds + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s)\, e_k, e_j\rangle_{-1} d\beta_k(s), \quad \forall\, j \in \mathbb{N}, \quad t \in [0,T].$$

We know also from [9] that for Ψ a maximal monotone multivalued function from $\mathbb{R}$ into $2^{\mathbb{R}}$ such that $0 \in \Psi(0)$ and

$$\sup\{|\theta| : \theta \in \Psi(r)\} \le C(1 + |r|^m), \qquad \forall r \in \mathbb{R}$$

under condition (3), for each $x \in L^p(\mathcal{O})$, $p \ge \max\{2m, 4\}$ there is a unique nonnegative solution $X \in L^\infty(0,T; L^p(\Omega; L^p(\mathcal{O})))$ to the equation (1).

In this work we are interested in the continuous dependence of the solution as function of Ψ for the stochastic porous media equation (1). This problem is relevant in asymptotic analysis and approximation of stochastic porous media equations.

To this propose we consider a family of maximal monotone graphs $\{\Psi^\alpha\}_{\alpha>0}$, Ψ and denote $A^\alpha = -\Delta\Psi^\alpha(x)$, with

$$D(A^\alpha) = \left\{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \Psi^\alpha(x) \in H_0^1(\mathcal{O})\right\}.$$

Suppose that the following assumptions are satisfied:

H$_1$ There exist some constants $m \ge 1$ and C independent of α such that

$$\sup\{|\theta| : \theta \in \Psi^\alpha(r)\} \le C(1 + |r|^m), \quad \forall \quad r \in \mathbb{R} \tag{5}$$

and

$$\sup\{|\theta| : \theta \in \Psi(r)\} \le C(1 + |r|^m), \qquad \forall r \in \mathbb{R}. \tag{6}$$

H$_2$ For all $\alpha > 0$ we have $0 \in \Psi^\alpha(0)$ and $0 \in \Psi(0)$.

H$_3$ We have $\Psi^\alpha \to \Psi$ as $\alpha \to 0$ in the graph sense, i. e.,

$$(1 + \lambda\Psi^\alpha)^{-1} x \longrightarrow (1 + \lambda\Psi)^{-1} x, \quad \forall \lambda > 0, \quad \forall x \in \mathbb{R}$$

for $\alpha \to 0$.

The main result is stated and proved in Section 2 and some examples are given in Section 3.

The following notations will be used throughout this paper.

$L^p(\mathcal{O})$, $p \ge 1$, is the usual space of $p-$integrable functions with norm denoted by $|\cdot|_p$. The scalar product in $L^2(\mathcal{O})$ and the duality induced by the space $L^2(\mathcal{O})$ will be denoted by $\langle \cdot, \cdot \rangle_2$.

For $p, q \in [1, +\infty]$ by $L_W^q(0,T; L^p(\Omega; H))$ (H a Hilbert space) we shall denote the space of all $q-$integrable processes $u : [0,T] \to L^p(\Omega; H)$ which are adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$.

By $C_W([0,T]; L^2(\Omega; H))$ we shall denote the space of all $H-$valued adapted processes which are mean square continuous (see [12], [13]).

This space is endowed with the norm

$$\|X\|^2_{C_W([0,T];L^2(\Omega,\mathcal{F},\mathbb{P};H))} = \sup_{t\in[0,T]} \mathbb{E}\,|X(t)|^2_H.$$

The main result (Theorem 2 below) amounts to saying that if $\Psi^\alpha \to \Psi$, for $\alpha \to 0$, then the solution X^α to (7) is convergent to the solution X to (1) and this may be seen as a Trotter type result for equation (1) (see e. g. [1], [3], [11] for corresponding deterministic results).

The Theorem 2 below is the main result of this paper.

2 The main result

Theorem 2 *Assume that H_1, H_2, H_3 and (3) hold. For each α consider the corresponding equations*

$$\left\{\begin{array}{lll} dX^{\alpha}(t)-\Delta\Psi^{\alpha}(X^{\alpha}(t))\,dt\ni\sigma(X^{\alpha}(t))\,dW(t), & in & (0,T)\times\mathcal{O} \\ \Psi(X(t))\ni 0, & on & (0,T)\times\partial\mathcal{O} \\ X(0)=x, & in & \mathcal{O}\end{array}\right. . \tag{7}$$

Then for each $x\in L^p(\mathcal{O})$, the corresponding solution X^{α} to (7) is convergent in

$$C_W\left([0,T];L^2\left(\Omega,\mathcal{F},\mathbb{P};H^{-1}(\mathcal{O})\right)\right)$$

for $\alpha\to 0$ to the solution X to (1), i. e.,

$$\lim_{\alpha\to 0}\mathbb{E}\left|X^{\alpha}(t)-X(t)\right|^2_{H^{-1}(\mathcal{O})}=0$$

uniformly on $[0,T]$.

Proof. Let X_λ be the solution to approximating equation

$$\left\{\begin{array}{lll} dX_\lambda(t)-\Delta\left(\Psi_\lambda(X_\lambda(t))+\lambda X_\lambda(t)\right)dt=\sigma(X_\lambda(t))\,dW(t), & \text{in} & (0,T)\times\mathcal{O} \\ X_\lambda(0)=x, & \text{in} & \mathcal{O}\end{array}\right. \tag{8}$$

where Ψ_λ is the Yosida approximation of Ψ, i. e.,

$$\Psi_\lambda(x)=\frac{1}{\lambda}\left(x-J_\lambda(x)\right)\in\Psi\left((1+\lambda\Psi)^{-1}(x)\right),\quad \lambda>0,\ x\in\mathbb{R}, \tag{9}$$

and $J_\lambda(x)=(1+\lambda\Psi)^{-1}(x)$. Note that $x\mapsto\Psi_\lambda(x)+\lambda x$ is strictly monotonically increasing.

Denote

$$\left\{\begin{array}{l} A_\lambda x=-\Delta\left(\Psi_\lambda(x)+\lambda x\right); \\ D(A_\lambda)=\left\{x\in H^{-1}(\mathcal{O})\cap L^1(\mathcal{O}):\Psi_\lambda(x)+\lambda x\in H_0^1(\mathcal{O})\right\}.\end{array}\right.$$

Consider also X_λ^α the corresponding solution to equation

$$\left\{\begin{array}{lll} dX_\lambda^\alpha(t)-\Delta\left(\Psi_\lambda^\alpha(X_\lambda^\alpha(t))+\lambda X_\lambda^\alpha(t)\right)dt=\sigma(X_\lambda^\alpha(t))\,dW(t), & \text{in} & (0,T)\times\mathcal{O} \\ X_\lambda^\alpha(0)=x, & \text{in} & \mathcal{O}\end{array}\right. \tag{10}$$

where Ψ_λ^α is the Yosida approximation of Ψ^α for each α.

Denote

$$\left\{\begin{array}{l} A_\lambda^\alpha x=-\Delta\left(\Psi_\lambda^\alpha(x)+\lambda x\right); \\ D(A_\lambda^\alpha)=\left\{x\in H^{-1}(\mathcal{O})\cap L^1(\mathcal{O}):\Psi_\lambda^\alpha(x)+\lambda x\in H_0^1(\mathcal{O})\right\}.\end{array}\right.$$

We have

$$\mathbb{E}\left|X^\alpha(t)-X(t)\right|_{-1}^2\le 3\Big(\mathbb{E}\left|X^\alpha(t)-X_\lambda^\alpha(t)\right|_{-1}^2+\mathbb{E}\left|X_\lambda^\alpha(t)-X_\lambda(t)\right|_{-1}^2$$
$$+\mathbb{E}\left|X_\lambda(t)-X(t)\right|_{-1}^2\Big).$$

By (6) we know from [[9] , (3.14)] that for $\lambda\to 0$ we have

$$(X_\lambda-X)\to 0\quad\text{strongly in}\quad L^2\left(\Omega;C\left([0,T];H^{-1}(\mathcal{O})\right)\right). \tag{11}$$

We shall prove now that as $\lambda\to 0$ we have

$$(X_\lambda^\alpha-X^\alpha)\to 0\quad\text{strongly in}\quad L^2\left(\Omega;C\left([0,T];H^{-1}(\mathcal{O})\right)\right) \tag{12}$$

uniformly in $\alpha > 0$.

Consider the section

$$\eta^{\alpha} \in \Psi^{\alpha}\left(X^{\alpha}\right), \quad a.e. \quad in \quad \Omega \times (0,T) \times \mathcal{O}$$

which arises in [7].

Using Ito's formula for equation

$$d\left(X^{\alpha}(t) - X^{\alpha}_{\lambda}(t)\right) - \Delta\left(\eta^{\alpha}(t) - \Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(t)\right) - \lambda X^{\alpha}_{\lambda}(t)\right) dt = \sigma\left(X^{\alpha}(t) - X^{\alpha}_{\lambda}(t)\right) dW(t)$$

with $\varphi(t,x) = |x|^2_{-1} e^{-\varepsilon t}$, we get that

$$\frac{1}{2}\left|X^{\alpha}(t) - X^{\alpha}_{\lambda}(t)\right|^2_{-1} e^{-\varepsilon t} + \int_0^t \int_{\mathcal{O}} \left[\eta^{\alpha}(s) - \Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right) - \lambda X^{\alpha}_{\lambda}(s)\right]\left(X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right) e^{-\varepsilon s} d\xi ds$$

$$\leq \int_0^t e^{-\varepsilon s}\left\langle X^{\alpha}(s) - X^{\alpha}_{\lambda}(s), \sigma\left(X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right) dW(s)\right\rangle_{-1}$$

$$+\left(-\frac{1}{2}\varepsilon\right)\left(\int_0^t \left|X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right|^2_{-1} e^{-\varepsilon s} ds\right)$$
$$+ c\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 \int_0^t \left|X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right|^2_{-1} e^{-\varepsilon s} ds, \quad \mathbb{P}\text{--}a.s..$$

By (9) we have $x = \lambda \Psi^{\alpha}_{\lambda}(x) + \left(1 + \lambda \Psi^{\alpha}\right)^{-1}(x)$ and this yields

$$\begin{aligned}
&\left[\eta^{\alpha}(s) - \Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right) - \lambda X^{\alpha}_{\lambda}(s)\right]\left(X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right) \\
&= \left(\eta^{\alpha}(s) - \Psi^{\alpha}\left(\left(1 + \lambda \Psi^{\alpha}\right)^{-1} X_{\lambda}(s)\right)\right)\left(X^{\alpha}(s) - \left(1 + \lambda \Psi^{\alpha}\right)^{-1} X^{\alpha}_{\lambda}(s)\right) \\
&\qquad - \lambda\left(\eta^{\alpha}(s) - \Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right)\right)\Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right) - \lambda X^{\alpha}_{\lambda}(s)\left(X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right) \\
&\geq \lambda\left(\left|\Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right)\right|^2 - \eta^{\alpha}(s)\Psi^{\alpha}_{\lambda}\left(X^{\alpha}_{\lambda}(s)\right)\right) + \lambda\left(\left|X^{\alpha}_{\lambda}(s)\right|^2 - X^{\alpha}_{\lambda}(s) X^{\alpha}(s)\right) \\
&\geq -\frac{\lambda}{4}\left|\eta^{\alpha}(s)\right|^2 - \frac{\lambda}{4}\left|X^{\alpha}(s)\right|^2, \quad \mathbb{P}\text{--}a.s.
\end{aligned}$$

using the monotonicity of Ψ^{α} and $\Psi^{\alpha}_{\lambda}(x) \in \Psi^{\alpha}\left(\left(1 + \lambda \Psi^{\alpha}\right)^{-1}(x)\right)$ for all $x \in \mathbb{R}$.

Hence for $\varepsilon > 0$ large enough we obtain for all $\lambda \in (0,1)$ and $t \in [0,T]$

$$\begin{aligned}
\frac{1}{2}\left|X^{\alpha}(t) - X^{\alpha}_{\lambda}(t)\right|^2_{-1} e^{-\varepsilon t} &\leq \frac{\lambda}{4}\int_0^t \int_{\mathcal{O}} \left(\left|\eta^{\alpha}(s)\right|^2 + \left|X^{\alpha}(s)\right|^2\right) d\xi ds \\
&\quad + \int_0^t e^{-\varepsilon s}\left\langle X^{\alpha}(s) - X^{\alpha}_{\lambda}(s), \sigma\left(X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right) dW(s)\right\rangle_{-1}.
\end{aligned} \tag{13}$$

We get for $\varepsilon > 0$, for all $\lambda \in (0,1)$, and $r \in [0,T]$ that

$$\begin{aligned}
\frac{1}{4}\mathbb{E}\sup_{t\in[0,r]}\left|X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right|^2_{-1} e^{-\varepsilon t} &\leq \frac{\lambda}{4}\mathbb{E}\int_0^r \int_{\mathcal{O}} \left(\left|\eta^{\alpha}(s)\right|^2 + \left|X^{\alpha}(s)\right|^2\right) d\xi ds \\
&\quad + c\mathbb{E}\left(\int_0^r \left|X^{\alpha}(s) - X^{\alpha}_{\lambda}(s)\right|^2_{-1} e^{-\varepsilon s} ds\right)
\end{aligned} \tag{14}$$

since by the Burkholder-Davis-Gundy inequality, we have

$$\int_0^t e^{-\varepsilon s} \langle X^\alpha(s) - X^\alpha_\lambda(s), \sigma(X^\alpha(s) - X^\alpha_\lambda(s))\, dW(s)\rangle_{-1}$$
$$\leq \mathbb{E}\left(c\int_0^r |X^\alpha(s) - X^\alpha_\lambda(s)|_{-1}^4 e^{-2\varepsilon s} ds\right)^{\frac{1}{2}}$$
$$\leq \mathbb{E} \sup_{s\in[0,r]} |X^\alpha(s) - X^\alpha_\lambda(s)|_{-1} e^{-\varepsilon s/2}\left(c\int_0^r |X^\alpha(s) - X^\alpha_\lambda(s)|_{-1}^2 e^{-\varepsilon s} ds\right)^{1/2}$$
$$\leq \frac{1}{4}\mathbb{E} \sup_{s\in[0,r]} |X^\alpha(s) - X^\alpha_\lambda(s)|_{-1}^2 e^{-\varepsilon s} + c\mathbb{E}\left(\int_0^r |X^\alpha(s) - X^\alpha_\lambda(s)|_{-1}^2 e^{-\varepsilon s} ds\right).$$

By the hypothesis $\mathbf{H}_1$ we have for all $x \in \mathbb{R}$ and all $\eta^\alpha \in \Psi^\alpha(x)$

$$|\eta^\alpha| \leq C(1 + |x|^m).$$

Consequently for $\eta^\alpha \in \Psi^\alpha(X^\alpha)$, $\quad a.e. \quad$ in $\quad \Omega \times (0,T) \times \mathcal{O}$ we get that

$$\frac{\lambda}{4}\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(|\eta^\alpha(s)|^2 + |X^\alpha(s)|^2\right) d\xi ds \leq \frac{\lambda}{4}C\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(|(1 + |X^\alpha(s)|^m)|^2 + |X^\alpha(s)|^2\right) d\xi ds$$
$$\leq \frac{\lambda}{4}C\left(1 + \mathbb{E}\int_0^t \int_{\mathcal{O}} |X^\alpha(s)|^p d\xi ds\right) \tag{15}$$

since $p \geq \max\{2m, 2\}$ and C is independent of λ and α.

We prove that

$$ess \sup_{t\in[0,T]} \mathbb{E}|X^\alpha_\lambda(t,x)|_p^p \leq \exp\left(c\frac{p-1}{2}\right)|x|_p^p \quad \forall \quad \lambda > 0, \quad \alpha > 0, \tag{16}$$

where $c > 0$ is independent of t, x, λ and α.

Note that relation (16) is similar to Lemma 3.1 from [9], but in the present paper we are interested to get c independent of α.

Indeed, for $A^\alpha_\lambda x = -\Delta(\Psi^\alpha_\lambda(x) + \lambda x)$, we take $(A^\alpha_\lambda)_\varepsilon$ the Yosida approximation of A^α_λ,

$$(A^\alpha_\lambda)_\varepsilon = \frac{1}{\varepsilon}\left(I - (I + \varepsilon A^\alpha_\lambda)^{-1}\right), \quad \varepsilon > 0$$

and we apply the Ito formula to

$$d(X^\alpha_\lambda)_\varepsilon(t) + (A^\alpha_\lambda)_\varepsilon (X^\alpha_\lambda)_\varepsilon(t)\, dt = \sigma((X^\alpha_\lambda)_\varepsilon(t))\, dW(t) \tag{17}$$

for the function $\varphi(x) = \frac{1}{p}|x|_p^p$. (More precisely we first apply Ito's formula to (17) for the function $\varphi_\gamma(x) = \frac{1}{p}\left|(1-\gamma\Delta)^{-1} x\right|_p^p$, $\gamma > 0$, and the we let $\gamma \to 0$. For more details see [[6], Lemma 3.5]).

We get

$$\mathbb{E}\varphi((X^\alpha_\lambda)_\varepsilon(t)) + \mathbb{E}\int_0^t \left\langle (A^\alpha_\lambda)_\varepsilon((X^\alpha_\lambda)_\varepsilon(s)), |(X^\alpha_\lambda)_\varepsilon(s)|^{p-2}(X^\alpha_\lambda)_\varepsilon(s)\right\rangle_2 ds \tag{18}$$
$$= \varphi(x) + \frac{p-1}{2}\sum_{k=1}^\infty \mu_k^2 \mathbb{E}\int_0^t \int_{\mathcal{O}} |(X^\alpha_\lambda)_\varepsilon(s)|^{p-2} |(X^\alpha_\lambda)_\varepsilon(s) e_k|^2 d\xi ds$$
$$\leq \varphi(x) + \frac{p-1}{2}c\mathbb{E}\int_0^t \int_{\mathcal{O}} |(X^\alpha_\lambda)_\varepsilon(s)|^p d\xi ds.$$

By [[6], (3.25)] , we have $\left|(Y_\lambda^\alpha)_\varepsilon\right|_p \le \left|(X_\lambda^\alpha)_\varepsilon\right|_p$ and this leads to

$$\left\langle (A_\lambda^\alpha)_\varepsilon (X_\lambda^\alpha)_\varepsilon , |(X_\lambda^\alpha)_\varepsilon|^{p-2} (X_\lambda^\alpha)_\varepsilon \right\rangle_2 = \frac{1}{\varepsilon} \left\langle (X_\lambda^\alpha)_\varepsilon - (Y_\lambda^\alpha)_\varepsilon , |(X_\lambda^\alpha)_\varepsilon|^{p-2} (X_\lambda^\alpha)_\varepsilon \right\rangle_2 \ge 0$$

where $(Y_\lambda^\alpha)_\varepsilon = (I + \varepsilon A_\lambda^\alpha)^{-1} (X_\lambda^\alpha)_\varepsilon$.

On another hand we have from [[6], Lemma 3.4]

$$\begin{aligned} (X_\lambda^\alpha)_\varepsilon &\to X_\lambda^\alpha \quad \text{strongly in} \quad L_W^\infty\left(0,T;L^2\left(\Omega;H^{-1}(\mathcal{O})\right)\right), \\ (X_\lambda^\alpha)_\varepsilon &\to X_\lambda^\alpha \quad \text{weak}^* \text{ in} \quad L_W^\infty\left(0,T;L^p\left(\Omega;L^p(\mathcal{O})\right)\right). \end{aligned}$$

Using Gronwall's lemma in (18) and letting ε tend to 0, we obtain (16) with $c > 0$ is independent of t, x, λ and α.

From [[9], (3.8)] we have for $\lambda \to 0$

$$X_\lambda^\alpha \to X^\alpha \quad \text{weak}^* \text{ in } L^\infty\left(0,T;L^p\left(\Omega;L^p(\mathcal{O})\right)\right).$$

Using [[10], Proposition III.12.] this yields

$$\begin{aligned} \underset{t\in[0,T]}{ess\sup} \mathbb{E}\left|X^\alpha(t,x)\right|_p^p \le & \liminf_\lambda \left(\underset{t\in[0,T]}{ess\sup} \mathbb{E}\left|X_\lambda^\alpha(t,x)\right|_p^p \right) \\ & \le \exp\left(c\frac{p-1}{2}\right)|x|_p^p \le C_1 |x|_p^p \end{aligned}$$

with $C_1 > 0$ is independent of t, x, λ and α.

Coming back to (15) we get that

$$\begin{aligned} \frac{\lambda}{4}\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(|\eta^\alpha(s)|^2 + |X^\alpha(s)|^2\right) d\xi ds &\le \frac{\lambda}{4} C \left(1 + \mathbb{E}\int_0^t \int_{\mathcal{O}} |X^\alpha(s)|^p d\xi ds\right) \\ &\le \frac{\lambda}{4} C_2 \left(1 + \underset{t\in[0,T]}{ess\sup} \mathbb{E}\left|X^\alpha(t,x)\right|_p^p\right) \\ &\le \frac{\lambda}{4} C_3 \left(1 + |x|_p^p\right) \end{aligned}$$

with $C_3 > 0$ is independent of t, x, λ and α.

Using Gronwall's lemma in (14) we get that

$$(X_\lambda^\alpha - X^\alpha) \to 0 \quad \text{strongly in} \quad L^2\left(\Omega; C\left([0,T]; H^{-1}(\mathcal{O})\right)\right)$$

for $\lambda \to 0$ uniformly in $\alpha > 0$.

In order to complete the proof it suffices to show that

$$(X_\lambda^\alpha - X_\lambda) \to 0 \quad \text{strongly in} \quad L^2\left(\Omega; C\left([0,T]; H^{-1}(\mathcal{O})\right)\right), \quad \forall \lambda > 0.$$

as $\alpha \to 0$.

Applying Ito's formula in equation

$$\begin{aligned} & d\left(X_\lambda^\alpha(t) - X_\lambda(t)\right) - \Delta\left(\Psi_\lambda^\alpha\left(X_\lambda^\alpha(t)\right) + \lambda X_\lambda^\alpha(t) - \Psi_\lambda\left(X_\lambda(t)\right) - \lambda X_\lambda(t)\right) dt \\ & = \sigma\left(X_\lambda^\alpha(t) - X_\lambda(t)\right) dW(t) \end{aligned}$$

with $\varphi(t,x) = |x|_{-1}^2 e^{-\varepsilon t}$ we have

$$\begin{aligned}
&\frac{1}{2}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 e^{-\varepsilon t} + \int_0^t \int_{\mathcal{O}} \left[\Psi_\lambda^\alpha\left(X_\lambda^\alpha(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda^\alpha(s) - X_\lambda(s)\right) e^{-\varepsilon s} d\xi ds \\
&+ \lambda \int_0^t \int_{\mathcal{O}} \left|X_\lambda^\alpha(s) - X_\lambda(s)\right|^2 e^{-\varepsilon s} d\xi ds \\
&\le \int_0^t e^{-\varepsilon s}\left\langle X_\lambda^\alpha(s) - X_\lambda(s), \sigma\left(X_\lambda^\alpha(s) - X_\lambda(s)\right) dW(s)\right\rangle_{-1} \\
&+ \left(-\frac{1}{2}\varepsilon\right)\left(\int_0^t \left|X_\lambda^\alpha(s) - X_\lambda(s)\right|_{-1}^2 e^{-\varepsilon s} ds\right) + c\sum_{k=1}^\infty \mu_k^2 \lambda_k^2 \int_0^t \left|X_\lambda^\alpha(s) - X_\lambda(s)\right|_{-1}^2 e^{-\varepsilon s} ds,
\end{aligned}$$

and for $\varepsilon > 0$, large enough,we get after some calculation involving the Burkholder-Davis-Gundy inequality, that

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\sup_{t\in[0,r]}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 e^{-\varepsilon t} \\
&\qquad + \mathbb{E}\int_0^r \int_{\mathcal{O}} \left[\Psi_\lambda^\alpha\left(X_\lambda^\alpha(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda^\alpha(s) - X_\lambda(s)\right) e^{-\varepsilon s} d\xi ds \\
&\le c\mathbb{E}\left(\int_0^r \left|X_\lambda^\alpha(s) - X_\lambda(s)\right|_{-1}^2 e^{-\varepsilon s} ds\right).
\end{aligned}$$

It is easily seen that

$$\left[\Psi_\lambda^\alpha\left(X_\lambda^\alpha(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda^\alpha(s) - X_\lambda(s)\right) \ge \left[\Psi_\lambda^\alpha\left(X_\lambda(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda^\alpha(s) - X_\lambda(s)\right),$$

$\mathbb{P}-a.s.$, since by the monotonicity of Ψ_λ we have that

$$\left(\Psi_\lambda^\alpha\left(X_\lambda^\alpha(s)\right) - \Psi_\lambda^\alpha\left(X_\lambda(s)\right)\right)\left(X_\lambda^\alpha(s) - X_\lambda(s)\right) \ge 0.$$

We obtain that

$$\frac{1}{4}\mathbb{E}\sup_{t\in[0,r]}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 e^{-\varepsilon t} \tag{19}$$

$$\begin{aligned}
&\le c\mathbb{E}\int_0^r \left|X_\lambda^\alpha(s) - X_\lambda(s)\right|_{-1}^2 e^{-\varepsilon s} ds \\
&+ \mathbb{E}\int_0^r \int_{\mathcal{O}} \left[\Psi_\lambda^\alpha\left(X_\lambda(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda(s) - X_\lambda^\alpha(s)\right) e^{-\varepsilon s} d\xi ds.
\end{aligned} \tag{20}$$

We have also that

$$\begin{aligned}
&\mathbb{E}\int_0^r \int_{\mathcal{O}} \left[\Psi_\lambda^\alpha\left(X_\lambda(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right]\left(X_\lambda(s) - X_\lambda^\alpha(s)\right) e^{-\varepsilon s} d\xi ds \\
&= \left\langle \Psi_\lambda^\alpha\left(X_\lambda(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right), \left(X_\lambda(s) - X_\lambda^\alpha(s)\right) e^{-\varepsilon s}\right\rangle_{L^2(\Omega\times[0,r]\times\mathcal{O})} \\
&\le \left|\Psi_\lambda^\alpha\left(X_\lambda(s)\right) - \Psi_\lambda\left(X_\lambda(s)\right)\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})} \left|\left(X_\lambda(s) - X_\lambda^\alpha(s)\right) e^{-\varepsilon s}\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})}.
\end{aligned}$$

Since $p > 2$ we have that

$$\begin{aligned}
&\left|\left(X_\lambda(s) - X_\lambda^\alpha(s)\right) e^{-\varepsilon s}\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})} \\
&\le C\left|X_\lambda\right|_{L^p(\Omega\times[0,r]\times\mathcal{O})} + C\left|X_\lambda^\alpha\right|_{L^p(\Omega\times[0,r]\times\mathcal{O})} \\
&\le C\left(\int_0^r \mathbb{E}\left|X_\lambda(s)\right|_{L^p(\mathcal{O})}^p ds\right)^{1/p} + C\left(\int_0^r \mathbb{E}\left|X_\lambda^\alpha(s)\right|_{L^p(\mathcal{O})}^p ds\right)^{1/p}
\end{aligned}$$

and by [[9], Lemma 3.1] and (16) we have

$$\left|\left(X_\lambda(s) - X_\lambda^\alpha(s)\right)e^{-\varepsilon s}\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})} \le C_4\left(1+|x|_p^p\right)^{1/p},$$

where C_4 is independent of x, t, λ and α.

On the other hand by $\mathbf{H}_1$ and [[9], Lemma 3.1] we have

$$\begin{aligned}
&\left|\Psi_\lambda^\alpha(X_\lambda(s)) - \Psi_\lambda(X_\lambda(s))\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})} \\
&\le \left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|\Psi_\lambda^\alpha(X_\lambda(s))\right|^2\right)d\xi ds\right)^{1/2} + \left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|\Psi_\lambda(X_\lambda(s))\right|^2\right)d\xi ds\right)^{1/2} \\
&\le C_5\left(\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(\left|1+|X_\lambda(s)|^m\right|^2\right)d\xi ds\right)^{1/2} \\
&\le C_6\left(1+|x|_p^p\right)^{1/2}.
\end{aligned} \tag{21}$$

with C_6 independent of x, t, λ and α.

Using $\mathbf{H}_3$, and

$$\left(\Psi_\lambda(X_\lambda(s)) - \Psi_\lambda^\alpha(X_\lambda(s))\right) = \frac{1}{\lambda}\left((1+\lambda\Psi)^{-1}X_\lambda(s) - (1+\lambda\Psi^\alpha)^{-1}X_\lambda(s)\right)$$

we get

$$\Psi_\lambda^\alpha(X_\lambda) \to \Psi_\lambda(X_\lambda) \text{ as } \alpha\to 0, \text{ a. e. on } \Omega\times[0,r]\times\mathcal{O}. \tag{22}$$

We obtain from (21) and (22) via the Lebesgue dominated convergence theorem that

$$\left|\Psi_\lambda^\alpha(X_\lambda(s)) - \Psi_\lambda(X_\lambda(s))\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})} \to 0 \text{ as } \alpha\to 0.$$

Gronwall's lemma applied to (19) leads to

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\sup_{t\in[0,r]}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 \\
&\le \left|\Psi_\lambda^\alpha(X_\lambda(s)) - \Psi_\lambda(X_\lambda(s))\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})}\left|\left(X_\lambda(s) - X_\lambda^\alpha(s)\right)e^{-\varepsilon s}\right|_{L^2(\Omega\times[0,r]\times\mathcal{O})}
\end{aligned}$$

and finally we get that

$$\mathbb{E}\sup_{t\in[0,r]}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 \to 0, \text{ as } \alpha\to 0, \quad \forall\lambda>0.$$

We can now come back to

$$\begin{aligned}
\mathbb{E}\left|X^\alpha(t) - X(t)\right|_{-1}^2 \le 3\Big(&\mathbb{E}\left|X^\alpha(t) - X_\lambda^\alpha(t)\right|_{-1}^2 + \mathbb{E}\left|X_\lambda^\alpha(t) - X_\lambda(t)\right|_{-1}^2 \\
&+\mathbb{E}\left|X_\lambda(t) - X(t)\right|_{-1}^2\Big).
\end{aligned}$$

Given $\varepsilon>0$ we first choose λ, independent of α, such that the first and the tierd terms are less then $\frac{\varepsilon}{3}$. Having fixed λ this way we can choose α such that the second term is less then $\frac{\varepsilon}{3}$ and finally we obtain

$$\mathbb{E}\left|X^\alpha(t) - X(t)\right|_{-1}^2 \le \varepsilon \text{ uniformly on } [0,T].$$

The proof of the main result is now complete. ■

3 Examples

1° Let $\Psi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ defined by

$$\Psi^{\alpha}(X) = |X|^{\alpha} \, signX, \quad 0 \leq \alpha < 1.$$

Equation (7) is called in this case the stochastic fast diffusion equation and is relevant in plasma physics (see [2]).

The case $\alpha = 0$ is relevant in stochastic models for self-organized criticality. The existence and longtime behaviors of solutions for equation were studied in [6], [7], [9], [14].

The extinction in finite time of solution for $0 < \alpha < 1$ was studied in [8].

As a consequence of Theorem 2 we obtain:

Corollary 3 *Consider the solution X^{α} to equation*

$$(23) \qquad \begin{cases} dX^{\alpha}(t) - \Delta\left(|X^{\alpha}(t)|^{\alpha} \, signX^{\alpha}(t)\right) dt \ni \sigma\left(X^{\alpha}(t)\right) dW(t), & in \quad (0,T) \times \mathcal{O} \\ |X^{\alpha}(t)|^{\alpha} \, signX^{\alpha}(t) \ni 0, & on \quad (0,T) \times \partial\mathcal{O} \\ X(0) = x, & in \quad \mathcal{O} \end{cases}.$$

Then for each $x \in L^p(\mathcal{O})$ and $\alpha \rightarrow 0$ the corresponding solution X^{α} to equation (23) is convergent in

$$C_W\left([0,T]; L^2\left(\Omega, \mathcal{F}, \mathbb{P}; H^{-1}(\mathcal{O})\right)\right)$$

to the solution X to equations

$$(24) \qquad \begin{cases} dX(t) - \Delta\left(signX(t)\right) dt \ni \sigma\left(X(t)\right) dW(t), & in \quad (0,T) \times \mathcal{O} \\ signX(t) \ni 0, & on \quad (0,T) \times \partial\mathcal{O} \\ X(0) = x, & in \quad \mathcal{O} \end{cases}$$

i. e.,

$$\mathbb{E}\left|X^{\alpha}(t) - X(t)\right|^2_{H^{-1}(\mathcal{O})} \rightarrow 0 \text{ uniformly on } [0,T] \text{ as } \alpha \rightarrow 0.$$

Proof.

It is easily seen that Ψ , $\Psi^{\alpha} : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ are maximal monotone graphs.

Since $\Psi^{\alpha}(X) = |X|^{\alpha} \, signX = |X|^{\alpha-1} X$ and $\alpha < 1 \leq m$ we have

$$\begin{aligned} \sup\left\{|\theta| : \theta \in \Psi^{\alpha}(X)\right\} &= \sup\left\{|\theta| : \theta \in |X|^{\alpha} \, signX\right\} \\ &\leq C\left(1 + |X|^{m}\right), \quad \forall \quad X \in \mathbb{R}. \end{aligned}$$

We also have that

$$\left(1 + \lambda\Psi^{\alpha}\right)^{-1} x \longrightarrow \left(1 + \lambda\Psi\right)^{-1} x, \quad \forall\lambda > 0, \quad \forall x \in \mathbb{R}$$

(for details see [1]).

The proof of the Corollary is now complete. ■

Remark 4 *The limit equation (24) is related to the model of self-organized criticality under stochastic perturbation (see [9]).*

2° The diffusivity function $\Psi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ of stochastic fast diffusion equation can also be written as

$$\Psi^{\alpha}(X) = |X|^{1-\alpha} \, signX, \quad 0 < \alpha \leq 1.$$

In case α is near 0, the corresponding equation can be regarded as a perturbation of stochastic heat equation.

By Theorem 2 we have that for each $x \in L^p(\mathcal{O})$ and $\alpha \to 0$ the solution X^α to equation

$$(25)\quad \begin{cases} dX^\alpha(t) - \Delta\left(|X^\alpha(t)|^{1-\alpha} \operatorname{sign} X^\alpha(t)\right) dt \ni \sigma(X^\alpha(t))\, dW(t), & \text{in} \quad (0,T)\times\mathcal{O} \\ |X^\alpha(t)|^{1-\alpha} \operatorname{sign} X^\alpha(t) \ni 0, & \text{on} \quad (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \text{in} \quad \mathcal{O} \end{cases},$$

is convergent in $C_W\left([0,T]; L^2\left(\Omega, \mathcal{F}, \mathbb{P}; H^{-1}(\mathcal{O})\right)\right)$ to the solution X to the linear stochastic heat equations

$$\begin{cases} dX(t) - \Delta X(t)\, dt = \sigma(X(t))\, dW(t), & \text{in} \quad (0,T)\times\mathcal{O} \\ X(t) = 0, & \text{on} \quad (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \text{in} \quad \mathcal{O} \end{cases},$$

i. e., $\mathbb{E}\left|X^\alpha(t) - X(t)\right|^2_{H^{-1}(\mathcal{O})} \to 0$ uniformly on $[0,T]$ as $\alpha \to 0$.

To conclude the second example, we just have to repeat the proof of the Corollary 3.

3° Let $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone graph of the form

$$\Psi(X) = \begin{cases} \Psi_1(X), & \text{if } X < a \\ [\Psi_1(a), \Psi_2(a)], & \text{if } X = a \\ \Psi_2(X), & \text{if } X > a \end{cases},$$

where Ψ_1 and Ψ_2 are continuous and monotone functions satisfying the assumption (6).

We define the approximation

$$\Psi^\alpha(X) = \begin{cases} \Psi_1(X), & \text{if } X < a-\alpha \\ \Psi_1(a-\alpha)\dfrac{a+\alpha-X}{2\alpha} + \Psi_2(a+\alpha)\dfrac{a-\alpha-X}{-2\alpha}, & \text{if } a-\alpha \le X \le a+\alpha \\ \Psi_2(X), & \text{if } a+\alpha < X \end{cases}.$$

Note that we have the approximation of a maximal monotone graph by continuous and monotone functions.

Using Theorem 2 we can prove the following corollary.

Corollary 5 *For each $x \in L^p(\mathcal{O})$ and $\alpha \to 0$ the corresponding solution X^α to equation*

$$(26)\quad \begin{cases} dX^\alpha(t) - \Delta\Psi^\alpha(X^\alpha(t))\, dt = \sigma(X^\alpha(t))\, dW(t), & \textit{in} \quad (0,T)\times\mathcal{O} \\ \Psi(X^\alpha(t)) = 0, & \textit{on} \quad (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \textit{in} \quad \mathcal{O} \end{cases},$$

is convergent in $C_W\left([0,T]; L^2\left(\Omega, \mathcal{F}, \mathbb{P}; H^{-1}(\mathcal{O})\right)\right)$ to the solution X of equations

$$\begin{cases} dX(t) - \Delta\Psi(X(t))\, dt \ni \sigma(X(t))\, dW(t), & \textit{in} \quad (0,T)\times\mathcal{O} \\ \Psi(X(t)) \ni 0, & \textit{on} \quad (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \textit{in} \quad \mathcal{O} \end{cases},$$

i. e., $\mathbb{E}\left|X^\alpha(t) - X(t)\right|^2_{H^{-1}(\mathcal{O})} \to 0$ uniformly on $[0,T]$ as $\alpha \to 0$.

Proof. Since Ψ_1 and Ψ_2 satisfies assumption (6) of $\mathbf{H}_1$ and for all $x \in \mathbb{R}$ we have $\lim_{\alpha\to 0} \Psi^{\alpha}(x) = \Psi(x)$, the proof of the Corollary 5 is immediate. ■

As a particular case we have the Heavside step function

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ [0,1], & \text{if } X = 0 \\ 1, & \text{if } X > 0 \end{cases},$$

which is relevant in the anomalous (singular) diffusion equation of the type

$$dX(t) = \Delta(H(X(t) - x_c))\,dt + \sigma(X(t))\,dW(t),$$

with x_c the critical value (see [9]).

Another particular case is

$$\Psi(X) = signX = \begin{cases} \frac{X}{|X|}, & \text{if } X \neq 0 \\ [-1,1], & \text{if } X = 0 \end{cases},$$

which as mentioned above is relevant in the stochastic models for self-organized criticality.

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