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Authors:

Teodor Havârneanu
"Al.I. Cuza" University and "Octav Mayer" Institute of Mathematics, Iaşi,Romania

Cătălin Popa
"Al.I. Cuza" University and "Octav Mayer" Institute of Mathematics, Iaşi,Romania

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INSTITUTE OF MATHEMATICS
"OCTAV MAYER"
ROMANIAN ACADEMY, IAŞI BRANCH
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On the equality between the variational and semigroupal solutions of a class of Hamilton-Jacobi equations*

Teodor Havârneanu
"Al.I. Cuza" University and "Octav Mayer "Institute of Mathematics,
Iaşi, Romania, email: havi@uaic.ro

Cătălin Popa
"Al.I. Cuza" University and "Octav Mayer "Institute of Mathematics,
Iaşi, Romania, email: cpopa@uaic.ro

Abstract

In this paper we prove the equality between the variational and semigroupal solutions of a class of Hamilton-Jacobi equations which were introduced and studied in [7].

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Key words: Hamilton-Jacobi equations, dynamic programming principle, semigroups of operators, Chernoff formula.

1 Introduction

In the paper [7] we have introduced and studied the variational and semigroupal solutions for the following Hamilton-Jacobi equation with man-min Hamiltonians

$$(1.1)\quad \begin{cases} U_t(t,x) - F(x, U_x(t,x)) - (Ax, U_x(t,x)) = g(x), & (t,x) \in R^+ \times H \\ U(0,x) = \varphi_0(x), & x \in H, \end{cases}$$

where H is a Hilbert space with the norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. The unknown function U is real valued and defined on $[0;+\infty) \times H$, φ_0, g are

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given functions on H and U_t, U_x stand for the derivatives with respect to variables t and x of the function U, respectively.

In the following we assume that A is the infinitesimal generator of a C_0-semigroup on H which satisfies $\|e^{At}\| \leq M$ and F is given by

$$F(x,p) = \min_{z\in Z} \max_{y\in Y}\{(f(y,z),p) + h(x,y,z)\}, \tag{1.2}$$

where Y, Z are two compact sets of a topological space Ω and $f : Y \times Z \to H$, $h : H \times Y \times Z \to R$, $g : H \to R$ are bounded and uniformly continuous functions with bound constants M_f, M_g, M_h and, moduli of uniform continuity $\omega_f, \omega_g, \omega_h$, respectively.

We also assume that the max-min condition is fulfilled, i.e.

$$\begin{aligned} &\min_{z\in Z} \max_{y\in Y}\{(f(y,z),p) + h(x,y,z)\} \\ &= \max_{y\in Y} \min_{z\in Z}\{(f(y,z),p) + h(x,y,z)\} \text{ for every } x,p \in H. \end{aligned} \tag{1.3}$$

Given a Banach space $(X, \|\cdot\|)$ denote by $BUC(X)$ the space of bounded uniformly continuous real valued functions on X endowed with the norm

$$\|f\|_b = \sup\{|f(x)|;\ x \in X\}.$$

By Lip(X) we denote the space of all Lipschitz functions $f : X \to R$.

It is well known that Eq. (1.1) is related to certain differential game [7,9]. The value of this differential game can be viewed as the generalized solution of Eq. (1.1).

Now we shall present the differential game. Consider the following sets

$$\begin{aligned} M(t) &= \{y : [t,+\infty) \to Y;\ y \text{ measurable}\}, \\ N(t) &= \{z : [t,+\infty) \to Z;\ z \text{ measurable}\}. \end{aligned} \tag{1.4}$$

$M(t)$ and $N(t)$ will be named sets of controls employed by players I and II, respectively.

Fix $t \geq 0, x \in H$ and consider the differential equation

$$\begin{cases} \dot{x}(s) = Ax(s) + f(y(s),z(s)),\ s \geq t, \\ x(t) = x, \end{cases} \tag{1.5}$$

where A and f satisfy the conditions from the previous section and $y \in M(t)$, $z \in N(t)$.

Following now [6, 8] we define the strategies of the player I (beginning at time t) as any mapping

$$\alpha : N(t) \to M(t)$$

such that for each $t \leq s$ and $z, \hat{z} \in N(t)$ we have

$$z(\tau) = \hat{z}(\tau) \quad \text{a.e. } t \leq \tau \leq s$$

implies

$$\alpha[z](\tau) = \alpha[\hat{z}](\tau) \quad \text{a.e. } t \leq \tau \leq s.$$

Similarly, any mapping

$$\beta : M(t) \to N(t)$$

with the property that for each $t \leq s$ and $y, \hat{y} \in M(t)$ satisfying $y(\tau) = \hat{y}(\tau)$ a.e. $t \leq \tau \leq s$ we have

$$\beta[y](\tau) = \beta[\hat{y}](\tau) \quad \text{a.e. } t \leq \tau \leq s,$$

is named a strategy of player II (beginning at time t).

We denote by $\Gamma(t)$ and $\Delta(t)$ the sets of all strategies beginning at time t for the player I and player II, respectively.

We associate with Eq. (2.2) the payoff functional

$$(\mathrm{P}_\lambda) \qquad P_\lambda(y, z) = \int_t^{+\infty} e^{-\lambda s} h(x(s), y(s), z(s)) ds,$$

where h satisfies the conditions from Section 1, $y \in M(t), z \in N(t)$, $x(t)$ is the “mild” solution of Eq. (2.2), and λ is a positive parameter.

The goal of player I is to maximize P_λ and the goal of player II is to minimize P_λ.

Using [7, Proposition 2.2, Lemma 2.1] (see also [1,2]) we define for every $y \in BUC(H)$ and $\lambda > 0$

$$(1.6) \qquad \begin{aligned} (R(\lambda)g)(x) &= \sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \left\{ \int_0^{+\infty} e^{-\lambda s} [g(x(s)) + h(x(s), \alpha[z](s), z(s))] ds \right\} \\ &= \inf_{\beta \in \Delta_0} \sup_{y \in M_0} \left\{ \int_0^{+\infty} e^{-\lambda s} [g(x(s)) + h(x(s), y(s), \beta[y](s))] ds \right\}, \end{aligned}$$

where $x(\cdot)$ from (1.6) is the solution of (1.5), where $y(\cdot)$ and $z(\cdot)$ are substituted by turn by $\alpha[z](\cdot)$ and $\beta[y](\cdot)$, respectively.

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset BUC(H) \to BUC(H)$ be the operator defined by (see [1])

$$\mathcal{A}R(1)g = R(1)g - g \text{ for each } g \in BUC(H) \tag{1.7}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \{\varphi = R(1)g;\ g \in BUC(H)\}.$$

Using now the Crandall-Ligget theorem (see [7]) we obtain that for each $\varphi_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $g \in BUC(H)$ the Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dt} \in \mathcal{A}\varphi + g \text{ in } R_+ \\ \varphi(0) = \varphi_0 \end{cases} \tag{1.8}$$

has a unique weak solution $\varphi : R^+ \to BUC(H)$ defined by the exponential formula

$$\varphi(t) = \lim_{n \to +\infty} \left(I - \frac{t}{n}\mathcal{A}\right)^{-n} \left(\frac{t}{n} g + \varphi_0\right).$$

The map

$$T(t) : \overline{\mathcal{D}(\mathcal{A})} \to \overline{\mathcal{D}(\mathcal{A})}$$

defined by

$$T(t)\varphi_0 = \varphi(t),\ t \geq 0$$

(where $\varphi(\cdot)$ is given by (1.8)) is a continuous semigroup of nonlinear contractions on $\overline{\mathcal{D}(\mathcal{A})}$ and it is called the semigroupal solution of Eq. (1.1).

Let us define the function

$$\begin{aligned} (S(t)\varphi_0)(x) &= \varphi(t,x) \\ &= \sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \left\{ \int_0^t [g(x(s)) + h(x(s), \alpha[z](s), z(s))]ds + \varphi_0(x(t)) \right\} \end{aligned} \tag{1.9}$$

where $x(\cdot)$ is the solution of (1.5) for $y(\cdot) = \alpha[z](\cdot)$.

By Proposition 3.2 ([7]), $S(t)$ is a semigroup of contractions on $BUC(H)$ and it is called the variational solution of Eq. (1.1).

2 The main result

The main result is contained in the following theorem:

Theorem 2.1. *If* $h, g \in BUC(H) \cap \mathrm{Lip}(H)$ *then*

$$S(t)\varphi_0 = T(t)\varphi_0 \text{ for all } t \geq 0 \text{ and } \varphi_0 \in \overline{\mathcal{D}(\mathcal{A})}.$$

Moreover, the operator $\mathcal{A}$ *is single valued and for all* $\varphi_0 \in \mathcal{D}(\mathcal{A})$

$$\lim \frac{1}{t}[(S(t)\varphi_0)(x) - \varphi_0(x)] = (\mathcal{A}\varphi_0)(x) + g(x), \text{ for } x \in H \tag{2.1}$$

and the limit in (2.1) *is in the strong topology of* H.

Proof. First of all we shall prove that $S(t)(\overline{\mathcal{D}(\mathcal{A})}) \subseteq \overline{\mathcal{D}(\mathcal{A})}$. Indeed, let $\varphi_0 \in \mathcal{E}$. Using Dynamic Programming Principle we have for all $0 \leq s \leq t$

$$\begin{aligned}(S(t)\varphi_0)(x) = &\sup_{\alpha \in \Gamma_0} \inf_{z \in N_0} \Big\{ \int_0^s [g(x(\tau)) + h(x(\tau), \alpha[z](\tau), z(\tau))] d\tau \\ &+ (S(t)\varphi_0)(x(t-s)); x(\cdot) \text{ verifies (1.5) with } g(\cdot) = \alpha[z](\cdot) \Big\}\end{aligned} \tag{2.2}$$

Let $\varepsilon > 0$. Then there exist $\alpha_\varepsilon \in \Gamma_0, z_\varepsilon \in N_0$ such that

$$\begin{aligned}&\int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))] d\tau + (S(t-s)\varphi_0)(x_\varepsilon(s)) - \varepsilon \\ &\leq (S(t)\varphi_0)(x) \leq \int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))] d\tau \\ &\qquad + (S(t-s)\varphi_0)(x_\varepsilon(s)) + \varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))] d\tau + (S(t-s)\varphi_0)(x_\varepsilon(s)) \\ &- (S(t)\varphi_0)(x_\varepsilon(s)) - \varepsilon \leq (S(t)\varphi_0)(x) - (S(t)\varphi_0)(x_\varepsilon(s)) \\ &\leq \int_0^s [g(x_\varepsilon(\tau)) + h(x_\varepsilon(\tau), \alpha_\varepsilon[z_\varepsilon](\tau), z_\varepsilon(\tau))] d\tau \\ &+ (S(t-s)\varphi_0)(x_\varepsilon(s)) - (S(t)\varphi_0)(x_\varepsilon(s)) + \varepsilon.\end{aligned}$$

Using the last relation we obtain

$$|(S(t)\varphi_0)(x) - (S(t)\varphi_0)(x_\varepsilon(s))| \leq cs + \|S(s)\varphi_0 - \varphi_0\|_b + \varepsilon \tag{2.3}$$

for certain positive constant c and $s \in (0;t)$.

Let $x(\cdot)$ be the solution of (1.5) with $g(\cdot) = \alpha[z](\cdot)$. Then we have

$$|(S(t)\varphi_0)(x_\varepsilon(s)) - (S(t)\varphi_0)(x)| \le c|x_\varepsilon(s) - x(s)| \le 2M_f cs, \text{ for } s \ge 0. \tag{2.4}$$

Using the fact that $\varphi_0 \in \mathcal{E}$ and the definition of $S(t)\varphi_0$ one can easily obtain

$$\|S(s)\varphi_0 - \varphi_0\| \le cs, \text{ for } s \ge 0. \tag{2.5}$$

From (2.3), (2.4) and (2.5) it results

$$|S(t)\varphi_0)(x(s)) - (S(t)\varphi_0)(x)| \le cs \text{ for } 0 \le s \le t.$$

Therefore $S(t)\varphi_0 \in \mathcal{E}$.

Using now Proposition 3.3 ([7]) we obtain the desired result.

Next, we give a nonlinear version of the Chernoff theorem (see [1]).

Proposition 2.1. *Let C be a closed convex subset of a Banach space Y and let $\mathcal{A}_0$ be a $m-$dissipative subset of $Y \times Y$. Let $\{G(t);\ t \ge 0\}$ be a family of nonexpansive mapping from C into itself such that*

$$\lim_{\rho \searrow 0} \left(I - \lambda \frac{G(\rho) - I}{\rho}\right)^{-1} x = (I - \lambda \mathcal{A}_0)^{-1} x \tag{2.6}$$

for all $x \in \overline{\mathcal{D}(\mathcal{A}_0)} \cap C$ and $\lambda > 0$. Then

$$\lim_{n \to +\infty} \left(G\left(\frac{t}{n}\right)\right)^n = e^{\mathcal{A}_0 t} x, \text{ for } t \ge 0,\ x \in \overline{\mathcal{D}(\mathcal{A}_0)} \cap C,$$

where $e^{\mathcal{A}_0 t}$ is the semigroup generated by $\mathcal{A}_0$.

We shall apply Proposition 2.1 with $C = \overline{\mathcal{E}}$, $G(\rho) = S(\rho)$ and $\mathcal{A}_0\varphi = \mathcal{A}\varphi + g$ for $\varphi \in \mathcal{D}(\mathcal{A})$. In this case the relation (2.6) becomes

$$\lim_{\rho \searrow 0} \left(I - \lambda \frac{S(\rho) - I}{\rho}\right)^{-1} \varphi_0 = (I - \lambda \mathcal{A}_0)^{-1}(\varphi_0 + \lambda g) \tag{2.7}$$

for $\varphi_0 \in \mathcal{E}$.

Using the nonexpansivity of $(I - \lambda\mathcal{A})^{-1}$ and $\left(I - \lambda \frac{S(\rho)-I}{\rho}\right)^{-1}$ on $\overline{\mathcal{E}} = \overline{\mathcal{D}(\mathcal{A})}$ we remark that it is sufficient to prove (2.7) for $\varphi_0 \in \mathcal{E}$.

We put

$$\varphi_\rho = \left(I - \lambda \frac{S(\rho) - I}{\rho}\right)^{-1} \varphi_0,\ \varphi = (I - \lambda\mathcal{A})^{-1}(\varphi_0 + \lambda g).$$

With these notations we have

$$\varphi_\rho = \frac{\rho}{\rho+\lambda}\varphi_0 + \frac{\lambda}{\rho+\lambda}S(\rho)\varphi_\rho. \tag{2.8}$$

Taking into account the definition of $S(\rho)$ we may write

$$\begin{aligned}\varphi_\rho(x) = \frac{\rho}{\rho+\lambda}\varphi_0(x) + \frac{\lambda}{\rho+\lambda}\sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\Big\{\int_0^\rho [g(x(t))\\ +h(x(t),\alpha[z](t),z(t))]dt + \varphi_\rho(x(\rho));\\ x(\cdot) \text{ verifies (1.5) with } g(\cdot)=\alpha[z](\cdot)\Big\}.\end{aligned} \tag{2.9}$$

Using the definition of $S(\rho)$ and the fact that $\varphi_0 \in \mathcal{E}$ we get

$$\|S(\rho)\varphi_0\|_{\mathrm{Lip}(H)} \le \rho M\|g\|_{\mathrm{Lip}(H)} + M\|\varphi_0\|_{\mathrm{Lip}(H)} \tag{2.10}$$

and

$$\|S(\rho)\varphi_0\|_b \le \rho(\|g\|_b + c) + \|\varphi_0\|_b, \tag{2.11}$$

for some positive constants c.

From (2.8), (2.10) and (2.11) it results

$$\|\varphi_\rho\|_{\mathrm{Lip}(H)} \le M\|\varphi_0\|_{\mathrm{Lip}(H)} + \lambda M\|g\|_{\mathrm{Lip}(H)}, \tag{2.12}$$

$$\|\varphi_\rho\|_b \le \|\varphi_0\|_b + \lambda(\|g\|_b + c). \tag{2.13}$$

Using now Theorem 2.1 ([7]) we obtain

$$\begin{aligned}\varphi(x) = \sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\Big\{\int_0^\rho e^{-\lambda^{-1}t}[g(x(t)) + \lambda^{-1}\varphi_0(x(t))\\ +h(x(t),\alpha[z](t),z(t))]dt + e^{-\lambda^{-1}\rho}\varphi(x(\rho));\ x(\cdot)\\ \text{solves (1.5) with } g(\cdot)=\alpha[z](\cdot)\Big\}.\end{aligned}$$

Therefore there exist $\alpha_\rho^1 \in \Gamma_0$, $z_\rho^1 \in N_0$ such that

$$\begin{aligned}\varphi_\rho(x) - \varphi(x) \le \frac{\rho}{\rho+\lambda}\varphi_0(x) + \frac{\lambda}{\rho+\lambda}\Big\{\int_0^\rho [g(x_\rho(t))\\ +h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))]dt + \varphi_\rho(x_\rho(\rho))\Big\}\\ -\int_0^\rho e^{-\lambda^{-1}t}[g(x_\rho(t)) + \lambda^{-1}\varphi_0(x_\rho(t))\\ +h(x_\rho(t),\alpha_\rho[z_\rho](t),z_\rho(t))]dt - e^{-\lambda^{-1}\rho}\varphi(x_\rho(\rho)) + 2\rho^2,\end{aligned} \tag{2.14}$$

where $x_\rho(\cdot)$ solves (1.5) for $\alpha = \alpha_\rho$, $y = \alpha[z](\cdot)$ and $z = z_\rho$.

In the same manner we get

$$
\begin{aligned}
\varphi(x) - \varphi_\rho(x) \le \frac{\rho}{\rho+\lambda}\varphi_0(x) + \frac{\lambda}{\rho+\lambda}\Big\{\int_0^\rho [g(\tilde{x}_\rho(t)) \\
+h(\tilde{x}_\rho(t), \tilde{\alpha}_\rho[\tilde{z}_\rho](t), \tilde{z}_\rho(t))]dt + \varphi_\rho(\tilde{x}(\rho))\Big\} \\
- \int_0^\rho e^{-\lambda^{-1}t}[g(\tilde{x}_\rho(t)) + \lambda^{-1}\varphi_0(\tilde{x}_\rho(t)) \\
+h(\tilde{x}_\rho(t), \tilde{\alpha}_\rho[\tilde{z}_\rho](t), \tilde{z}_\rho(t))]dt - e^{-\lambda^{-1}\rho}\varphi_0(\tilde{x}_\rho(\rho)) + 2\rho^2,
\end{aligned} \tag{2.15}
$$

for some $\tilde{\alpha}_\rho \in \Gamma_0$ and $\tilde{z}_\rho \in N_0$.

From (2.14) we obtain

$$
\begin{aligned}
\varphi_\rho(x) - \varphi(x) \le \left|\frac{\rho}{\rho+\lambda}\varphi_0(x) - \frac{1}{\lambda}\int_0^\rho e^{-\lambda^{-1}t}\varphi_0(x_\rho(t))dt\right| \\
+\left|\frac{\lambda}{\rho+\lambda}\int_0^\rho g(x_\rho(t))dt - \int_0^\rho e^{-\lambda^{-1}t} g(x_\rho(t))dt\right| \\
+\left|\frac{\lambda}{\rho+\lambda}\int_0^\rho h(x_\rho(t), \alpha_\rho[z_\rho](t), z_\rho(t))dt \right.\\
\left. - \int_0^\rho e^{-\lambda^{-1}t} h(x_\rho(t), \alpha_\rho[z_\rho](t), z_\rho(t))dt\right| + \left|\frac{\lambda}{\rho+\lambda}\varphi_\rho(x_\rho(\rho)) \right.\\
\left. -e^{-\lambda^{-1}\rho}\varphi(x_\rho(\rho))\right| + 2\rho^2 \le \int_0^\rho \left|\frac{1}{\rho+\lambda}\varphi_0(x) - \frac{1}{\lambda}e^{-\lambda^{-1}t}\varphi_0(x_\rho(t))\right| dt \\
+\int_0^\rho \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}t}\right| (|g(x_\rho(t))| + |h(x_\rho(t), \alpha_\rho[z_\rho](t), z_\rho(t))|)dt \\
+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho - \varphi\|_b + \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}\rho}\right| \|\varphi\|_b + 2\rho^2 \\
\le \int_0^\rho \left|\frac{1}{\rho+\lambda}\varphi_0(x) - \frac{1}{\rho+\lambda}\varphi_0(x_\rho(t))\right| dt \\
+\int_0^\rho \left|\frac{1}{\rho+\lambda} - \frac{e^{-\lambda^{-1}t}}{\lambda}\right| |\varphi_0(x_\rho(t))| dt \\
+\int_0^\rho \left|\frac{\lambda}{\rho+\lambda} - e^{-\lambda^{-1}t}\right| (M_g + M_h)dt
\end{aligned}
$$

$$+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho-\varphi\|_b+\left|\frac{\lambda}{\rho+\lambda}-e^{-\lambda^{-1}\rho}\right|\|\varphi\|_b+2\rho^2$$

$$\leq \frac{1}{\rho+\lambda}\int_0^\rho ctdt+\int_0^\rho\left|\frac{1}{\rho+\lambda}-\frac{e^{-\lambda^{-1}t}}{\lambda}\right|\|\varphi_0\|_b dt$$

$$+\int_0^\rho\left|\frac{\lambda}{\rho+\lambda}-e^{-\lambda^{-1}t}\right|(M_g+M_h)dt$$

$$+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho-\varphi\|_b+\left|\frac{\lambda}{\rho+\lambda}-e^{-\lambda^{-1}\rho}\right|\|\varphi_0\|_b+2\rho^2$$

$$=a(\rho)+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho-\varphi\|_b.$$

Analogously, using (2.15) we obtain

$$\varphi(x)-\varphi_\rho(x)\leq a(\rho)+\frac{\lambda}{\rho+\lambda}\|\varphi_\rho-\varphi\|_b,\ \forall x\in H.$$

Therefore

$$\frac{\rho}{\rho+\lambda}\|\varphi-\varphi_\rho\|_b\leq a(\rho),\ \ \rho>0.$$

Since $\frac{a(\rho)}{\rho}\underset{\rho\to 0}{\longrightarrow}0$, we get

$$\lim_{\rho\to 0}\|\varphi-\varphi_\rho\|_b=0.$$

Applying Proposition 2.1 we have

$$T(t)\varphi_0=\lim_{n\to+\infty}\left(S\left(\frac{t}{n}\right)\right)^n\varphi_0=S(t)\varphi_0,$$

for every $\varphi_0\in\overline{\mathcal{E}}=\overline{\mathcal{D}(\mathcal{A})}$, $t\geq 0$.

Now we shall prove (2.1).

Let $\varepsilon>0$. Then there exists $\alpha_\varepsilon\in\Gamma_0$, such that

$$(S(t)\varphi_0)(x)-\varphi_0(x)\leq\int_0^t[g(x_\varepsilon^1(\tau))$$
$$+h(x_\varepsilon^1(\tau),\alpha_\varepsilon[z](\tau),z(\tau))]d\tau+\varphi_0(x_\varepsilon^1(t))-\varphi_0(x)+\varepsilon, \tag{2.16}$$

for every $z\in N_0$ and $x_\varepsilon^1(\cdot)$ solves (1.5) for $\alpha=\alpha_\varepsilon$, $y=\alpha[z](\cdot)$ and $z(\cdot)$.

Using now Proposition 2.2 ([7]) and since $\varphi_0\in\mathcal{D}(\mathcal{A})$ there exists $f\in BUC(H)$ such that

$$\varphi_0(x)=\sup_{\alpha\in\Gamma_0}\inf_{z\in N_0}\left\{\int_0^t[f(x(\tau))+h(x(\tau),\alpha[z](\tau),z(\tau))]d\tau+e^{-t}\varphi_0(x(t))\right\}.$$

Therefore, there exists $z_\varepsilon \in \Gamma_0$ such that

$$\varphi_0(x) \geq \left\{ \int_0^t [f(x_\varepsilon^2(\tau)) + h(x_\varepsilon^2(\tau), \alpha[z_\varepsilon](\tau), z_\varepsilon(\tau))] d\tau + e^{-t} \varphi_0(x_\varepsilon^2(t)) \right\} - \varepsilon, \tag{2.17}$$

for every $\alpha \in \Gamma_0$.

Taking $\varepsilon = t^2$ and using (2.16) and (2.17) it results

$$(S(t)\varphi_0)(x) - \varphi_0(x) \leq \int_0^t g(x_{t^2}(\tau)) d\tau$$

$$+ \int_0^t (1 - e^{-\tau}) h(x_{t^2}(\tau), \alpha_{t^2}[z_{t^2}](\tau), z_{t^2}(\tau)) d\tau$$

$$+ (1 - e^{-t}) \varphi_0(x_{t^2}(t)) - \int_0^t e^{-t} f(x_{t^2}(\tau)) d\tau + 2t^2.$$

Dividing now the last relation by t and making $t \to 0$ we obtain

$$\lim_{t \to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) \leq g(x) - f(x) + \varphi_0(x).$$

Similarly we get

$$\lim_{t \to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) \geq g(x) - f(x) + \varphi_0(x).$$

Therefore

$$\lim_{t \to 0_+} \frac{1}{t}((S(t)\varphi_0)(x) - \varphi_0(x)) = \varphi_0(x) - f(x) + g(x)$$

$$= (R(1)f)(x) - f(x) + g(x) = (\mathcal{A}(R(1)f))(x)$$

$$+ g(x) = (\mathcal{A}\varphi_0)(x) + g(x), \text{ for every } x \in H.$$

The proof of Theorem 2.1 is finished.

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